EINIGE
SÄTZE ÜBER DIE FUNCTIONEN $C_n^\nu(x)$

VON

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Ich werde in den folgenden Zeilen eine Reihe von neuen Theoremen aus der Theorie der Functionen $C_n^\nu(x)$ mittheilen. Im ersten Capitel stelle ich verschiedene Integralausdrücke für die Functionen $C_n^\nu(x)$ und $D_n^\nu(x)$ auf, mit deren Hilfe sodann im zweiten Abschnitte mehrere nach den Functionen $C_n^\nu(x)$ fortschreitende Reihen als bestimmte Integrale dargestellt und einige höchst merkwürdige Darstellungen des Jacobi-Legendre'schen Symbols durch vielfache bestimmte Integrale abgeleitet werden. Im dritten Capitel werden hierauf verschiedene Relationen ermittelt, die sich theils auf die Functionen $C_n^\nu(x)$, theils auf Producte von zwei solchen Functionen beziehen, und einige Eigenschaften der Coëfficienten dieser Functionen, sowie ein Satz über das Zeichen eines gewissen von ihnen abhängigen bestimmten Integrales angegeben, endlich im vierten Capitel verschiedene zahlentheoretische Functionen mit Hilfe der Functionen $C_n^\nu(x)$ durch vielfache Integrale ausgedrückt und aus diesen Darstellungen die asymptotischen Werthe einiger der erwähnten Integrale abgeleitet.

1. Da die Functionen $C_n^\nu(x)$ und $D_n^\nu(x)$ specielle hypergeometrische Reihen sind und daher mit Hilfe der Gauss'schen und Kummer'schen Umformungen in mannigfacher Weise durch solche Reihen ausgedrückt werden können, so ergeben sich aus jeder, gewisse Bedingungen erfüllenden Integraldarstellung der allgemeinen oder von geeigneten speciellen hypergeometrischen Reihen ein oder mehrere Integralausdrücke für die allgemeinen oder eventuell specielle Functionen $C_n^\nu(x)$ und $D_n^\nu(x)$.

So liefern z. B. die von mir aufgestellten allgemeinen Formeln

$$\int_0^\infty e^{-ax} x^\rho J^\mu(a_1 x) = \frac{\Pi(\rho+\mu) a_1^\mu}{2^\mu \Pi(\mu) a^{\rho+\mu+1}} F\left(\frac{\rho+\mu+1}{2}, \frac{\rho+\mu}{2}+1, \mu+1, -\frac{a_1^2}{a^2}\right) {}^{1)} \qquad (|R(a)|>|J(a_1)|)$$

[1]) „Über einige bestimmte Integrale". Sitzungsberichte der kais. Akademie der Wissenschaften, mathem.-naturw. Classe, 72. Band, II. Abtheilung.

$$1a)\quad \int_{-1}^{+1} \frac{(1-x^2)^{\frac{2\nu-1}{2}} C_n^\nu(x)\,dx}{(1-cx)^\mu} =$$

$$= \frac{\Pi(\nu-1)\,\Pi(\mu+n-1)\,\Pi(n+2\nu-1)}{2^{n+1}\,\Pi(n+\nu)\,\Pi(n)\,\Pi(\mu-1)} \left[\frac{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(2\nu-1)}\right]^2 c^n F\left(\frac{n+\mu}{2},\ \frac{n+\mu+1}{2},\ n+\nu+1,\ c^2\right)^{1)} \qquad (|c|<1)$$

für die Function

$$1)\qquad D_n^\nu(x) = \sum_{\lambda=0}^{\lambda=\infty} \frac{\Pi(n+2\nu+2\lambda-1)}{2^{n+2\lambda+1}\,\Pi(\lambda)\,\Pi(n+\nu+\lambda)\,x^{n+2\nu+2\lambda}}$$

$$= \frac{\Pi(n+2\nu-1)}{2^{n+1}\,\Pi(n+\nu)\,x^{n+2\nu}}\, F\left(\frac{n}{2}+\nu,\ \frac{n+1}{2}+\nu,\ n+\nu+1,\ \frac{1}{x^2}\right)$$

die Integralausdrücke

$$D_n^\nu(x) = \frac{z^{\nu-1}}{i^{n+\nu}} \int_0^\infty e^{-xz} z^{\nu-1} J^{n+\nu}(iz)\,dz \qquad (|x|>1)$$

$$D_n^\nu(x) = \frac{\Pi(n)\,\Pi(2\nu-1)}{\Pi(n+2\nu-1)\,\Pi(\nu-1)} \left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1} \frac{(1-z^2)^{\frac{2\nu-1}{2}} C_n^\nu(z)\,dz}{(x-z)^{2\nu}} \qquad (|x|>1).$$

Es ergibt sich ferner aus der Verbindung des Euler'schen Integrales für die hypergeometrische Reihe

$$F(\alpha, \beta, \gamma, x^2) = \frac{2\,\Pi(\gamma-1)}{\Pi(\alpha-1)\,\Pi(\gamma-\alpha-1)} \int_0^1 z^{2\alpha-1}(1-z^2)^{\gamma-\alpha-1}(1-x^2z^2)^{-\beta}\,dz$$

in welchem x^2 nicht auf der Strecke $1 \ldots \infty$ liegen darf, mit einer der Relationen

$$2)\qquad C_n^\nu(x) = \frac{\Pi(n+\nu-1)}{\Pi(n)\,\Pi(\nu-1)}\, \xi^n F(\nu,\ -n,\ -n-\nu+1,\ \xi^{-2})$$

$$3)\qquad D_n^\nu(x) = \frac{2^{2\nu-1}\,\Pi(\nu-1)\,\Pi(n+2\nu-1)}{\Pi(n+\nu)}\, \xi^{-(n+2\nu)} F(\nu,\ n+2\nu,\ n+\nu+1,\ \xi^{-2})$$

$$2x = \xi + \xi^{-1}$$

für die Functionen $C_n^\nu(x)$ bez. $D_n^\nu(x)$ die Formel

$$3a)\qquad C_n^\nu(x) = \frac{\Pi(n+2\nu-1)}{2^{2\nu-1}[\Pi(\nu-1)]^2\,\Pi(n)} \int_0^\pi (x \pm \cos\varphi\,\sqrt{x^2-1})^n \sin^{2\nu-1}\varphi\,d\varphi$$

$$D_n^\nu(x) = (-i)^{2\nu-1} \frac{\Pi(n+2\nu-1)}{\Pi(n)} \int_0^\infty \frac{(\sin it)^{2\nu-1}\,dt}{(x+\sqrt{x^2-1}\,\cos it)^{n+2\nu}}$$

$$D_n^\nu(x) = (-1)^{2\nu} i^{n+2\nu} \frac{\Pi(n+2\nu-1)}{\Pi(n)} \int_0^{\log\sqrt{\frac{x+1}{x-1}}} (x-\sqrt{x^2-1}\,\cos iv)^n (\sin iv)^{2\nu-1}\,dv.$$

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[1] „Zur Theorie der Functionen $C_n^\nu(x)$." Diese Denkschriften, 48. Band.

Andere Integralausdrücke fliessen aus der von den Herren Sonine[1]) und P. Schafheitlin[2]) mitgetheilten interessanten Integraldarstellung der hypergeometrischen Reihe, welche unter dem Integralzeichen ein Product von zwei Bessel'schen Functionen erster Art enthält. Dieselbe ist übrigens nichts anderes, als eine einfache Umformung des eben angeführten Euler'schen Integrales. Ersetzt man nämlich in demselben den Factor $(1-x^2z^2)^{-\beta}$ mit Hilfe der von Herrn Sonine a. a. O. aufgestellten Relation

$$\int_0^\infty J^n(bz)J^m(az)\,z^{m-n+1}dz = \frac{b^m(a^2-b^2)^{n-m-1}}{2^{n-m-1}\Pi(n-m-1)a^n} \qquad (a>b;\; n>m>-1)$$

durch ein bestimmtes Integral, so entstehen nach Umkehrung der Integrationsordnung die Relationen

$$F(\alpha,\beta,\gamma,x^2) = \frac{\Pi(\gamma-1)\Pi(-\beta)}{2^{\beta-1}\Pi(\alpha-1)\Pi(\gamma-\alpha-1)x^m}\int_0^\infty J^{m+1-\beta}(y)\,y^\beta dy\int_0^1 z^{2\alpha-m-1}(1-z^2)^{\gamma-\alpha-1}J^m(xyz)\,dz \qquad (|x|<1)$$

$$F\left(\alpha,\alpha-\gamma+1,\alpha-\beta+1,\frac{1}{x^2}\right) =$$

$$= \frac{\Pi(\alpha-\beta)\,\Pi(\gamma-\alpha-1)x^m}{2^{\alpha-\gamma}\Pi(\alpha-1)\Pi(-\beta)}\int_0^\infty J^{m+\gamma-\alpha}(y)\,y^{\alpha-\gamma+1}dy\int_0^1 z^{2\alpha-m-1}(1-z^2)^{-\beta}J^m\left(\frac{yz}{x}\right)dz \qquad (|x|>1).$$

Multiplicirt man die Gleichung

$$J^m(\rho yz) = \frac{(\rho yz)^m}{2^m}\sum_{\lambda=0}^{\lambda=\infty}(-1)^\lambda\frac{(\rho yz)^{2\lambda}}{2^{2\lambda}\Pi(\lambda)\Pi(m+\lambda)}$$

mit $z^{2\alpha-m-1}(1-z^2)^\mu dz$, integrirt von $z=0$ bis $z=1$ und berücksichtigt, dass

4) $$\int_0^1 z^{2(\alpha+\lambda)-1}(1-z^2)^\mu dz = \frac{\Pi(\alpha+\lambda-1)\Pi(\mu)}{2\Pi(\alpha+\lambda+\mu)}$$

ist, so erhält man die Beziehung

$$\int_0^1 J^m(\rho yz)z^{2\alpha-m-1}(1-z^2)^\mu dz = \frac{\Pi(\mu)(\rho y)^m}{2^{m+1}}\sum_{\lambda=0}^{\lambda=\infty}(-1)^\lambda\frac{\Pi(\alpha+\lambda-1)(\rho y)^{2\lambda}}{2^{2\lambda}\Pi(\lambda)\Pi(\alpha+\lambda+\mu)\Pi(\lambda+m)},$$

welche für $\alpha=m+1$ in die von Herrn Sonine a. a. O. auf anderem Wege ermittelte specielle Formel

$$\int_0^1 J^{\alpha-1}(\rho yz)z^\alpha(1-z^2)^\mu dz = \frac{2^\mu\Pi(\mu)}{(\rho y)^{\mu+1}}J^{\mu+\alpha}(\rho y) \qquad (\mu>-1;\; \alpha-1 \text{ und } \mu+\alpha \text{ keine negativen ganzen Zahlen})$$

übergeht.

Mit Hilfe dieser Relation verwandelt man nun die letzten Gleichungen sofort in die folgenden

5) $$F(\alpha,\beta,\gamma,x^2) = \frac{2^{\gamma-\alpha-\beta}\Pi(\gamma-1)\Pi(-\beta)}{\Pi(\alpha-1)x^{\gamma-1}}\int_0^\infty\frac{J^{\alpha-\beta}(y)J^{\gamma-1}(xy)\,dy}{y^{\gamma-\alpha-\beta}} \qquad (|x|<1)$$

6) $$F\left(\alpha,\alpha-\gamma+1,\alpha-\beta+1,\frac{1}{x^2}\right) = \frac{2^{\gamma-\alpha-\beta}\Pi(\gamma-\alpha-1)\Pi(\alpha-\beta)}{\Pi(\alpha-1)x^{\gamma-2\alpha-1}}\int_0^\infty\frac{J^{\alpha-\beta}(y)J^{\gamma-1}(xy)\,dy}{y^{\gamma-\alpha-\beta}} \qquad (|x|>1).$$

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[1]) „Recherches sur les fonctions cylindriques." Mathematische Annalen von F. Klein, 16. Band.

[2]) „Über die Darstellung der hypergeometrischen Reihe durch ein bestimmtes Integral". Mathematische Annalen von F. Klein, 30. Band.

wo weder $\alpha-\beta$ noch $\gamma-1$ eine negative ganze Zahl ist, α und $\gamma-\alpha-\beta+1$ grösser als 0 und $\beta<1$ sein muss. Von der letzten beschränkenden Bedingung kann man sich übrigens durch ein bekanntes Verfahren leicht freimachen, so dass man auf diesem Wege rasch zur erwähnten Integraldarstellung der hypergeometrischen Reihe gelangt.

Durch die Verbindung der Gleichungen 1) und 6) und der bekannten Definitionsgleichung der Functionen $C_n^\nu(x)$

$$C_n^\nu(x)=\sum_{\mu=0}^{\mu=\left[\frac{n}{2}\right]}(-1)^{\left[\frac{n}{2}\right]-\mu}\frac{\Pi\left(n+\mu-\left[\frac{n}{2}\right]+\nu-1\right)(2x)^{n-2\left[\frac{n}{2}\right]+2\mu}}{\Pi\left(\left[\frac{n}{2}\right]-\mu\right)\Pi\left(n-2\left[\frac{n}{2}\right]+2\mu\right)\Pi(\nu-1)}$$

$$=(-1)^{\left[\frac{n}{2}\right]}\frac{2\Pi\left(n+\nu-\left[\frac{n}{2}\right]-1\right)x^{n-2\left[\frac{n}{2}\right]}}{\Pi\left(\left[\frac{n}{2}\right]\right)\Pi(\nu-1)}F\left(n+\nu-\left[\frac{n}{2}\right],-\left[\frac{n}{2}\right],n-2\left[\frac{n}{2}\right]+\frac{1}{2},x^2\right)$$

mit 4) ergeben sich für die Functionen $C_n^\nu(x)$ und $D_n^\nu(x)$ folgende Integralausdrücke

$$C_{2r}^\nu(x)=\frac{(-1)^r}{2^{\nu-1}\Pi(\nu-1)}\int_0^\infty y^{\nu-1}J^{2r+\nu}(y)\cos xy\,dy\qquad\left(|x|<1,\nu<\frac{3}{2}\right)$$

$$C_{2r+1}^\nu(x)=\frac{(-1)^r}{2^{\nu-1}\Pi(\nu-1)}\int_0^\infty y^{\nu-1}J^{2r+\nu+1}(y)\sin xy\,dy\qquad\left(|x|<1,\nu<\frac{3}{2}\right)$$

$$D_n^\nu(x)=\frac{\Pi(n+2\nu-1)\Pi\left(-\frac{n+1}{2}-\nu\right)\sqrt{\pi}}{2^{n+\nu+1}\Pi\left(\frac{n}{2}+\nu-1\right)}\int_0^\infty y^{\nu-1}J^{n+\nu}(y)\cos xy\,dy\qquad\left(|x|>1,\nu<\frac{3}{2}\right)$$

$$D_n^\nu(x)=\frac{\Pi(n+2\nu-1)\Pi\left(-\frac{n}{2}-\nu\right)\sqrt{\pi}}{2^{n+\nu+1}\Pi\left(\frac{n-1}{2}+\nu\right)}\int_0^\infty y^{\nu-1}J^{n+\nu}(y)\sin xy\,dy\qquad\left(|x|>1,\nu<\frac{3}{2}\right).$$

Auf demselben Wege ergeben sich aus den Gleichungen 2) und 3) die Relationen

$$C_n^\nu(x)=\frac{\Pi(n+\nu-1)\Pi(-n-\nu)\xi^\nu}{2^{2\nu-1}[\Pi(\nu-1)]^2}\int_0^\infty y^{2\nu-1}J^{-n-\nu}(y)J^{n+\nu}(\xi y)\,dy\qquad(1>\nu;\ \xi+\xi^{-1}=2x;\ |\xi|>1)$$

$$D_n^\nu(x)=\Pi(n+2\nu-1)\Pi(-n-2\nu)\xi^\nu\int_0^\infty J^{n+\nu}(y)J^{-n-\nu}(\xi y)y^{2\nu-1}\,dy\qquad(1>\nu;\ \xi+\xi^{-1}=2x;\ |\xi|>1)$$

$$D_n^\nu(x)=\Pi(\nu-1)\Pi(-\nu)\xi^\nu\int_0^\infty y^{2\nu-1}J^{n+\nu}(y)J^{n+\nu}(\xi y)\,dy\qquad(1>\nu;\ \xi+\xi^{-1}=2x;\ |\xi|>1).$$

Um eine neue Darstellung der Functionen $C_n^\nu(x)$ durch eine hypergeometrische Reihe und daraus einen neuen Integralausdruck derselben Kategorie für dieselbe abzuleiten, transformirt man die lineare Differentialgleichung zweiter Ordnung

$$(1-x^2)y''-(2\nu+1)xy'+n(n+2\nu)y=0$$

deren vollständiges Integral bekanntlich

$$y=a\,C_n^\nu(x)+b\,D_n^\nu(x)$$

ist, durch die Substitution

$$\frac{1-x}{2} = z$$

wodurch dieselbe in die specielle hypergeometrische Differentialgleichung

$$z(1-z)y'' + \frac{2\nu+1}{2}(1-2z)y' + n(n+2\nu)y = 0$$

übergeführt wird mit dem allgemeinen Integrale

$$y = \alpha F\left(n+2\nu, -n, \frac{2\nu+1}{2}, z\right) + \beta \frac{1}{(x-x^2)^{\nu-\frac{1}{2}}} \int_0^1 \frac{y^{n+\nu-\frac{1}{2}}(1-y)^{n+\nu-\frac{1}{2}}}{(x-y)^{n+1}} dy\,.$$

Man hat daher die Relation

$$C_n^\nu(x) = \frac{\Pi(n+2\nu-1)}{\Pi(n)\Pi(2\nu-1)} F\left(n+2\nu, -n, \frac{2\nu+1}{2}, \frac{1-x}{2}\right)$$

oder

$$C_n^\nu(\cos x) = \frac{\Pi(n+2\nu-1)}{\Pi(n)\Pi(2\nu-1)} F\left(n+2\nu, -n, \frac{2\nu+1}{2}, \sin^2\frac{x}{2}\right) \tag{7}$$

$$= (-1)^n \frac{\Pi(n+2\nu-1)}{\Pi(n)\Pi(2\nu-1)} F\left(n+2\nu, -n, \frac{2\nu+1}{2}, \cos^2\frac{x}{2}\right)$$

deren Verbindung mit 5) unmittelbar die Formeln

$$C_n^\nu(\cos x) = \frac{\Pi\left(\frac{2\nu-1}{2}\right)}{2^{\frac{2\nu-1}{2}}\Pi(2\nu-1)\sin^{\frac{2\nu-1}{2}}\frac{x}{2}} \int_0^\infty J^{2(n+\nu)}(y) J^{\frac{2\nu-1}{2}}\left(y\sin\frac{x}{2}\right) y^{\frac{2\nu-1}{2}} dy \qquad \left(\nu > -\frac{1}{2}\right)$$

$$C_n^\nu(\cos x) = (-1)^n \frac{\Pi\left(\frac{2\nu-1}{2}\right)}{2^{\frac{2\nu-1}{2}}\Pi(2\nu-1)\cos^{\frac{2\nu-1}{2}}\frac{x}{2}} \int_0^\infty J^{2(n+\nu)}(y) J^{\frac{2\nu-1}{2}}\left(y\cos\frac{x}{2}\right) y^{\frac{2\nu-1}{2}} dy \qquad \left(\nu > -\frac{1}{2}\right)$$

liefert.

Um einen Integralausdruck für die Function $C_n^\nu(x)$ zu erhalten, welcher unter dem Integralzeichen die Function $C_m^\mu(x)$ enthält multiplicirt man die Gleichung

$$C_n^{2\nu}\left(z\sin\frac{x}{2}\right) = \sum_{\mu=0}^{\mu=\left[\frac{n}{2}\right]} (-1)^{\left[\frac{n}{2}\right]-\mu} \frac{\Pi\left(n+2\nu+\mu-\left[\frac{n}{2}\right]-1\right)\left(2z\sin\frac{x}{2}\right)^{n-2\left[\frac{n}{2}\right]+2\mu}}{\Pi\left(\left[\frac{n}{2}\right]-\mu\right)\Pi\left(n-2\left[\frac{n}{2}\right]+2\mu\right)\Pi(2\nu-1)}$$

mit $z^{n-2\left[\frac{n}{2}\right]}(1-z^2)^\rho dz$, und integrirt von $z=0$ bis $z=1$, wodurch wegen 4) die Relation

$$\int_0^1 C_n^{2\nu}\left(z\sin\frac{x}{2}\right) z^{n-2\left[\frac{n}{2}\right]}(1-z^2)^\rho dz =$$

$$= (-1)^{\left[\frac{n}{2}\right]} \frac{\Pi(\rho)\Pi\left(n+2\nu-\left[\frac{n}{2}\right]-1\right)\sqrt{\pi}}{2\Pi(\nu-1)\Pi\left(\left[\frac{n}{2}\right]\right)\Pi\left(n-2\left[\frac{n}{2}\right]+\rho+\frac{1}{2}\right)} \sin\frac{x}{2}^{n-2\left[\frac{n}{2}\right]} F\left(n+\nu-\left[\frac{n}{2}\right], -\left[\frac{n}{2}\right], n-2\left[\frac{n}{2}\right]+\rho+\frac{3}{2}, \sin^2\frac{x}{2}\right)$$

entsteht.

Setzt man

$$\rho = \nu - 1 + \left[\frac{n}{2}\right] - \frac{n}{2}$$

so wird nach 7) die auf der rechten Seite dieser Gleichung stehende hypergeometrische Reihe gleich $C^{\nu+\frac{n}{2}-\left[\frac{n}{2}\right]}_{\left[\frac{n}{2}\right]}(\cos x)$ und daher hat man die bemerkenswerthen Formeln

7 *a*)
$$C^{\nu+\frac{n}{2}-\left[\frac{n}{2}\right]}_{\left[\frac{n}{2}\right]}(\cos x) =$$
$$= \frac{(-1)^{\left[\frac{n}{2}\right]} 2^{2\left[\frac{n}{2}\right]-n+1}}{\sqrt{\pi}\sin^{n-2\left[\frac{n}{2}\right]}\frac{x}{2}} \left(\frac{\Pi\left(\frac{2\nu-1}{2}-n+2\left[\frac{n}{2}\right]\right)}{\Pi(\nu-1)}\right)^{(-1)^{n-2\left[\frac{n}{2}\right]}} \int_0^1 C_n^{2\nu}\left(z\sin\frac{x}{2}\right) z^{n-2\left[\frac{n}{2}\right]}\left(1-z^2\right)^{\nu-1+\left[\frac{n}{2}\right]-\frac{n}{2}} dz$$

$$C^{\nu+\frac{n}{2}-\left[\frac{n}{2}\right]}_{\left[\frac{n}{2}\right]}(\cos x) =$$
$$= \frac{2^{2\left[\frac{n}{2}\right]-n+1}}{\sqrt{\pi}\cos^{n-2\left[\frac{n}{2}\right]}\frac{x}{2}} \left(\frac{\Pi\left(\frac{2\nu-1}{2}-n+2\left[\frac{n}{2}\right]\right)}{\Pi(\nu-1)}\right)^{(-1)^{n-2\left[\frac{n}{2}\right]}} \int_0^1 C_n^{2\nu}\left(z\cos\frac{x}{2}\right) z^{n-2\left[\frac{n}{2}\right]}\left(1-z^2\right)^{\nu-1+\left[\frac{n}{2}\right]-\frac{n}{2}} dz,$$

oder

$$C_r^{\nu}(\cos x) = (-1)^r \frac{2\Pi\left(\frac{2\nu-1}{2}\right)}{\sqrt{\pi}\,\Pi(\nu-1)} \int_0^1 C_{2r}^{2\nu}\left(z\sin\frac{x}{2}\right)\left(1-z^2\right)^{\nu-1} dz$$
$$= \frac{2\Pi\left(\frac{2\nu-1}{2}\right)}{\sqrt{\pi}\,\Pi(\nu-1)} \int_0^1 C_{2r}^{2\nu}\left(z\cos\frac{x}{2}\right)\left(1-z^2\right)^{\nu-1} dz$$

$$C_r^{\nu+\frac{1}{2}}(\cos x) = (-1)^r \frac{\Pi(\nu-1)}{\sqrt{\pi}\,\Pi\left(\frac{2\nu-3}{2}\right)\sin\frac{x}{2}} \int_0^1 C_{2r+1}^{2\nu}\left(z\sin\frac{x}{2}\right)\left(1-z^2\right)^{\nu-\frac{3}{2}} dz$$
$$= \frac{\Pi(\nu-1)}{\sqrt{\pi}\,\Pi\left(\frac{2\nu-3}{2}\right)\cos\frac{x}{2}} \int_0^1 C_{2r+1}^{2\nu}\left(z\cos\frac{x}{2}\right)\left(1-z^2\right)^{\nu-\frac{3}{2}} dz.$$

Setzt man in diesen Formeln

$$z\sin\frac{x}{2}, \quad \text{bez.} \quad z\cos\frac{x}{2} = \sin\frac{\varphi}{2}$$

und schreibt in der letzten von ihnen für $\nu : \nu - \frac{1}{2}$, so erhält man die Gleichungen

$$C_r^{\nu}(\cos x) = (-1)^r \frac{\Pi\left(\frac{2\nu-1}{2}\right)}{\sqrt{\pi}\,\Pi(\nu-1)\sin^{2\nu-1}\frac{x}{2}} \int_0^x \left(\sin^2\frac{x}{2} - \sin^2\frac{\varphi}{2}\right)^{\nu-1} C_{2r}^{2\nu}\left(\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}\, d\varphi$$
$$= \frac{(-1)^r\,\Pi\left(\frac{2\nu-1}{2}\right)}{2^{\nu-1}\sqrt{\pi}\,\Pi(\nu-1)\sin^{2\nu-1}\frac{x}{2}} \int_0^x (\cos\varphi - \cos x)^{\nu-1} C_{2r}^{2\nu}\left(\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}\, d\varphi$$

$$C_r^\nu(\cos x) = \frac{\Pi\left(\frac{2\nu-1}{2}\right)}{\sqrt{\pi}\,\Pi(\nu-1)\cos^{2\nu-1}\frac{x}{2}} \int_0^{\pi-x} \left(\cos^2\frac{x}{2} - \sin^2\frac{\varphi}{2}\right)^{\nu-1} C_{2r}^{2\nu}\left(\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}\,d\varphi$$

$$= \frac{\Pi\left(\frac{2\nu-1}{2}\right)}{2^{\nu-1}\sqrt{\pi}\,\Pi(\nu-1)\cos^{2\nu-1}\frac{x}{2}} \int_x^{\pi} (\cos x - \cos\varphi)^{\nu-1} C_{2r}^{2\nu}\left(\cos\frac{\varphi}{2}\right)\sin\frac{\varphi}{2}\,d\varphi$$

$$C_r^\nu(\cos x) = \frac{(-1)^r\,\Pi\left(\frac{2\nu-3}{2}\right)}{2^{\nu-1}\sqrt{\pi}\,\Pi(\nu-2)\sin^{2\nu-1}\frac{x}{2}} \int_0^{x} (\cos\varphi - \cos x)^{\nu-2} C_{2r+1}^{2\nu-1}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\,d\varphi$$

$$= \frac{(-1)^r\,\Pi\left(\frac{2\nu-3}{2}\right)}{2\sqrt{\pi}\,\Pi(\nu-2)\sin^{2\nu-1}\frac{x}{2}} \int_0^{x} \left(\sin^2\frac{x}{2} - \sin^2\frac{\varphi}{2}\right)^{\nu-2} C_{2r+1}^{2\nu-1}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\,d\varphi$$

$$C_r^\nu(\cos x) = \frac{\Pi\left(\frac{2\nu-3}{2}\right)}{2\sqrt{\pi}\,\Pi(\nu-2)\cos^{2\nu-1}\frac{x}{2}} \int_0^{\pi-x} \left(\cos^2\frac{x}{2} - \sin^2\frac{\varphi}{2}\right)^{\nu-2} C_{2r+1}^{2\nu-1}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\,d\varphi$$

$$= \frac{\Pi\left(\frac{2\nu-3}{2}\right)}{2^{\nu-1}\sqrt{\pi}\,\Pi(\nu-2)\cos^{2\nu-1}\frac{x}{2}} \int_x^{\pi} (\cos x - \cos\varphi)^{\nu-2} C_{2r+1}^{2\nu-1}\left(\cos\frac{\varphi}{2}\right)\sin\varphi\,d\varphi.$$

Berücksichtigt man, dass

$$C_{2n}^1(\cos\chi) = \frac{\sin(2n+1)\chi}{\sin\chi}$$

$$C_{2n}^1(\sin\chi) = (-1)^n\frac{\cos(2n+1)\chi}{\cos\chi}$$

ist, so erhält man aus diesen Gleichungen die folgenden von Herrn Mehler[1]) im 5. Bande der „Mathematischen Annalen“ aufgestellten Integralausdrücke für die Kugelfunctionen erster Art

$$P_r(\cos x) = \frac{2}{\pi}\int_0^x \frac{\cos\left(r+\frac{1}{2}\right)\varphi\,d\varphi}{\sqrt{2(\cos\varphi - \cos x)}}$$

$$P_r(\cos x) = \frac{2}{\pi}\int_x^{\pi} \frac{\sin\left(r+\frac{1}{2}\right)\varphi\,d\varphi}{\sqrt{2(\cos x - \cos\varphi)}}.$$

Mit Hilfe der bekannten Relationen

8) $$C_n^\mu(x) - C_{n-2}^\mu(x) = \frac{n+\mu-1}{\mu-1}\,C_n^{\mu-1}(x)$$

9) $$n\,C_n^\mu(x) + (n+2\mu-2)\,C_{n-2}^\mu(x) = 2(n+\mu-1)\,x\,C_{n-1}^\mu(x)$$

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[1]) „Notiz über die Dirichlet'schen Integralausdrücke für die Kugelfunction $P^n(\cos\vartheta)$ und eine analoge Integralform für die Cylinderfunction $J(x)$.“

kann man die eben abgeleiteten Gleichungen in die folgenden umformen

10)
$$C_r^\nu(\cos x)=\frac{(-1)^r\,\Pi\left(\frac{2\nu-1}{2}\right)}{2^{\nu-1}\sqrt{\pi}\,\Pi(\nu-1)\sin^{2\nu-1}\frac{x}{2}}\left\{\frac{2r+2\nu-1}{2\nu-1}\int_0^x(\cos\varphi-\cos x)^{\nu-1}C_{2r}^{2\nu-1}\left(\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}\,d\varphi+\int_0^x(\cos\varphi-\cos x)^{\nu-1}C_{2r-2}^{2\nu}\left(\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}\,d\varphi\right\}$$

$$C_r^\nu(\cos x)=\frac{\Pi\left(\frac{2\nu-1}{2}\right)}{2^{\nu-1}\sqrt{\pi}\,\Pi(\nu-1)\cos^{2\nu-1}\frac{x}{2}}\left\{\frac{2r+2\nu-1}{2\nu-1}\int_x^\pi(\cos x-\cos\varphi)^{\nu-1}C_{2r}^{2\nu-1}\left(\cos\frac{\varphi}{2}\right)\sin\frac{\varphi}{2}\,d\varphi+\int_x^\pi(\cos x-\cos\varphi)^{\nu-1}C_{2r-2}^{2\nu}\left(\cos\frac{\varphi}{2}\right)\sin\frac{\varphi}{2}\,d\varphi\right\}$$

11)
$$C_r^\nu(\cos x)=\frac{(-1)^r\,\Pi\left(\frac{2\nu-1}{2}\right)}{2^{\nu}r\sqrt{\pi}\,\Pi(\nu-1)\sin^{2\nu-1}\frac{x}{2}}\left\{(2r+2\nu-1)\int_0^x(\cos\varphi-\cos x)^{\nu-1}C_{2r-1}^{2\nu}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\,d\varphi-2(r+2\nu-1).\int_0^x(\cos\varphi-\cos x)^{\nu-1}C_{2r-2}^{2\nu}\left(\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}\,d\varphi\right\}$$

$$C_r^\nu(\cos x)=\frac{\Pi\left(\frac{2\nu-1}{2}\right)}{2^{\nu}r\sqrt{\pi}\,\Pi(\nu-1)\cos^{2\nu-1}\frac{x}{2}}\left\{(2r+2\nu-1)\int_x^\pi(\cos x-\cos\varphi)^{\nu-1}C_{2r-1}^{2\nu}\left(\cos\frac{\varphi}{2}\right)\sin\varphi\,d\varphi-2(r+2\nu-1).\int_x^\pi(\cos x-\cos\varphi)^{\nu-1}C_{2r-2}^{2\nu}\left(\cos\frac{\varphi}{2}\right)\sin\frac{\varphi}{2}\,d\varphi\right\}$$

12)
$$C_r^\nu(\cos x)=\frac{(-1)^r\,\Pi\left(\frac{2\nu-3}{2}\right)}{2^{\nu-1}\sqrt{\pi}\,\Pi(\nu-2)\sin^{2\nu-1}\frac{x}{2}}\left\{\frac{2r+2\nu-1}{2(\nu-1)}\int_0^x(\cos\varphi-\cos x)^{\nu-2}C_{2r+1}^{2\nu-2}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\,d\varphi+\int_0^x(\cos\varphi-\cos x)^{\nu-2}C_{2r-1}^{2\nu-1}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\,d\varphi\right\}$$

$$C_r^\nu(\cos x)=\frac{\Pi\left(\frac{2\nu-3}{2}\right)}{2^{\nu-1}\sqrt{\pi}\,\Pi(\nu-2)\cos^{2\nu-1}\frac{x}{2}}\left\{\frac{2r+2\nu-1}{2(\nu-1)}\int_x^\pi(\cos x-\cos\varphi)^{\nu-2}C_{2r+1}^{2\nu-2}\left(\cos\frac{\varphi}{2}\right)\sin\varphi\,d\varphi+\int_x^\pi(\cos x-\cos\varphi)^{\nu-2}C_{2r-1}^{2\nu-1}\left(\cos\frac{\varphi}{2}\right)\sin\varphi\,d\varphi\right\}$$

13)
$$C_r^\nu(\cos x)=\frac{(-1)^r\,\Pi\left(\frac{2\nu-3}{2}\right)}{2^{\nu}r\sqrt{\pi}\,\Pi(\nu-2)\sin^{2\nu-1}\frac{x}{2}}\left\{2(2r+2\nu-1).\int_0^x(\cos\varphi-\cos x)^{\nu-2}C_{2r}^{2\nu-1}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\sin\frac{\varphi}{2}\,d\varphi-(2r+4\nu-3)\int_0^x(\cos\varphi-\cos x)^{\nu-2}C_{2r-1}^{2\nu-1}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\,d\varphi\right\}$$

$$13) \quad C_r^\nu(\cos x) = \frac{\Pi\left(\frac{2\nu-3}{2}\right)}{2^\nu r\sqrt{\pi}\,\Pi(\nu-2)\cos^{2\nu-1}\frac{x}{2}}\Bigg\{2(2r+2\nu-1).$$

$$\cdot\int_x^\pi(\cos x-\cos\varphi)^{\nu-2}C_{2r}^{2\nu-1}\left(\cos\frac{\varphi}{2}\right)\sin\varphi\cos\frac{\varphi}{2}\,d\varphi-(2r+4\nu-3)\int_x^\pi(\cos x-\cos\varphi)^{\nu-2}C_{2r-1}^{2\nu-1}\left(\cos\frac{\varphi}{2}\right)\sin\varphi\,d\varphi\Bigg\}.$$

Berücksichtigt man, dass in jedem dieser Gleichungspaare die zweiten Integrale auf der rechten Seite bis auf das Zeichen $(-1)^{r-1}$ gleich sind, so erhält man durch Addition der zwei Gleichungen jedes Paares die vier neuen Formeln

$$C_r^\nu(\cos x) = \frac{\Pi\left(\frac{2\nu-3}{2}\right)(2r+2\nu-1)}{2^{\nu+1}\sqrt{\pi}\,\Pi(\nu-1)}\Bigg\{\frac{(-1)^r}{\sin^{2\nu-1}\frac{x}{2}}\int_0^x(\cos\varphi-\cos x)^{\nu-1}C_{2r}^{2\nu-1}\left(\cos\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}\,d\varphi+\frac{1}{\cos^{2\nu-1}\frac{x}{2}}.$$

$$\cdot\int_x^\pi(\cos x-\cos\varphi)^{\nu-1}C_{2r}^{2\nu-1}\left(\cos\frac{\varphi}{2}\right)\sin\frac{\varphi}{2}\,d\varphi\Bigg\}$$

$$C_r^\nu(\cos x) = \frac{\Pi\left(\frac{2\nu-1}{2}\right)(2r+2\nu-1)}{2^{\nu+1}r\sqrt{\pi}\,\Pi(\nu-1)}\Bigg\{\frac{(-1)^r}{\sin^{2\nu-1}\frac{x}{2}}\int_0^x(\cos\varphi-\cos x)^{\nu-1}C_{2r-1}^{2\nu}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\,d\varphi+\frac{1}{\cos^{2\nu-1}\frac{x}{2}}.$$

$$\cdot\int_x^\pi(\cos x-\cos\varphi)^{\nu-1}C_{2r-1}^{2\nu}\left(\cos\frac{\varphi}{2}\right)\sin\varphi\,d\varphi\Bigg\}$$

$$C_r^\nu(\cos x) = \frac{\Pi\left(\frac{2\nu-3}{2}\right)(2r+2\nu-1)}{2^{\nu+1}\sqrt{\pi}\,\Pi(\nu-1)}\Bigg\{\frac{(-1)^r}{\sin^{2\nu-1}\frac{x}{2}}\int_0^x(\cos\varphi-\cos x)^{\nu-2}C_{2r+1}^{2\nu-2}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\,d\varphi+\frac{1}{\cos^{2\nu-1}\frac{x}{2}}.$$

$$\cdot\int_x^\pi(\cos x-\cos\varphi)^{\nu-2}C_{2r+1}^{2\nu-2}\left(\cos\frac{\varphi}{2}\right)\sin\varphi\,d\varphi\Bigg\}$$

$$C_r^\nu(\cos x) = \frac{\Pi\left(\frac{2\nu-3}{2}\right)(2r+2\nu-1)}{2\,r\sqrt{\pi}\,\Pi(\nu-2)}\Bigg\{\frac{(-1)^r}{\sin^{2\nu-1}\frac{x}{2}}\int_0^x(\cos\varphi-\cos x)^{\nu-2}C_{2r}^{2\nu-1}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\sin\frac{\varphi}{2}\,d\varphi+\frac{1}{\cos^{2\nu-1}\frac{x}{2}}.$$

$$\cdot\int_x^\pi(\cos x-\cos\varphi)^{\nu-2}C_{2r}^{2\nu-1}\left(\cos\frac{\varphi}{2}\right)\sin\varphi\cos\frac{\varphi}{2}\,d\varphi\Bigg\}.$$

Bedenkt man, dass

$$\left[\frac{1}{\mu}C_n^\mu(\cos\chi)\right]_{\mu=0} = \frac{2}{n}\cos n\chi$$

ist, so erhält man aus diesen Gleichungen die bekannten von Dirichlet im 17. Bande des Crelle'schen Journales angegebenen Integrale für die Kugelfunctionen erster Art

$$\frac{\pi}{2}P_r(\cos x) = \int_0^x\frac{\cos r\varphi\cos\frac{\varphi}{2}\,d\varphi}{\sqrt{2(\cos\varphi-\cos x)}}+\int_x^\pi\frac{\cos r\varphi\sin\frac{\varphi}{2}\,dx}{\sqrt{2(\cos x-\cos\varphi)}}$$

$$\frac{\pi}{2}P_r(\cos x) = -\int_0^x\frac{\sin r\varphi\sin\frac{\varphi}{2}\,d\varphi}{\sqrt{2(\cos\varphi-\cos x)}}+\int_x^\pi\frac{\sin r\varphi\cos\frac{\varphi}{2}\,d\varphi}{\sqrt{2(\cos x-\cos\varphi)}}.$$

Der in den vorstehenden Zeilen zur Ableitung dieser Formeln benützte directe Weg ist zweifellos auch der einfachste von allen, welche zu diesen Relationen führen.

Ein ferneres Integral für die Functionen $C_n^\nu(x)$ ergibt sich in folgender Weise. Multiplicirt man den Zähler und Nenner des λ^{ten} Gliedes der auf der rechten Seite der Gleichung

$$C_n^\nu(x) = \sum_{\lambda=0}^{\lambda=\left[\frac{n}{2}\right]} (-1)^\lambda \frac{\Pi(n+\nu-\lambda-1)(2x)^{n-2\lambda}}{\Pi(\lambda)\Pi(n-2\lambda)\Pi(\nu-1)}$$

stehenden Summe mit $\Pi\left(\frac{2\lambda-1}{2}\right)$ und berücksichtigt, dass

$$2^{2\lambda}\Pi(\lambda)\Pi\left(\frac{2\lambda-1}{2}\right) = \sqrt{\pi}\,\Pi(2\lambda)$$

ist, so erhält man die Relation

$$C_n^\nu(x) = \frac{2^n}{\sqrt{\pi}\,\Pi(\nu-1)} \sum_{\lambda=0}^{\lambda=\left[\frac{n}{2}\right]} (-1)^\lambda \frac{\Pi(n+\nu-\lambda-1)\,\Pi\left(\frac{2\lambda-1}{2}\right)}{\Pi(n-2\lambda)\Pi(2\lambda)} x^{n-2\lambda}.$$

Nun ist bekanntlich

$$\frac{\Pi(n+\nu-\lambda-1)\,\Pi\left(\frac{2\lambda-1}{2}\right)}{2\Pi\left(n+\nu-\frac{1}{2}\right)} = \int_0^{\frac{\pi}{2}} \cos^{2n+2\nu-2\lambda-1}\varphi\,\sin^{2\lambda}\varphi\,d\varphi$$

und daher verwandelt sich diese Relation in

$$C_n^\nu(x) = \frac{2^{n+1}\Pi\left(n+\nu-\frac{1}{2}\right)}{\sqrt{\pi}\,\Pi(\nu-1)} \int_0^{\frac{\pi}{2}} \cos^{n+2\nu-1}\varphi\,d\varphi \sum_{\lambda=0}^{\lambda=\left[\frac{n}{2}\right]} (-1)^\lambda \frac{\sin^{2\lambda}\varphi\,(x\cos\varphi)^{n-2\lambda}}{\Pi(n-2\lambda)\Pi(2\lambda)}.$$

Die auf der rechten Seite dieser Gleichung stehende Summe ist aber der reelle Bestandtheil von $\frac{1}{\Pi(n)}(x\cos\varphi+i\sin\varphi)^n$ und daher hat man die Relationen

$$14)\qquad C_n^\nu(x) = \frac{2^n\Pi\left(n+\nu-\frac{1}{2}\right)}{\Pi(n)\,\Pi(\nu-1)\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \cos^{n+2\nu-1}\varphi\,\{(x\cos\varphi+i\sin\varphi)^n+(x\cos\varphi-i\sin\varphi)^n\}\,d\varphi$$

$$= \frac{2^n\Pi\left(n+\nu-\frac{1}{2}\right)}{\Pi(n)\,\Pi(\nu-1)\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \sin^{n+2\nu-1}\varphi\,\{(x\sin\varphi+i\cos\varphi)^n+(x\sin\varphi-i\cos\varphi)^n\}\,d\varphi.$$

2. Die Vereinigung der in dem vorigen Capitel abgeleiteten Ausdrücke für die Functionen $C_n^\nu(x)$ mit Relationen, welche ich in einer früheren Mittheilung[1]) aufgestellt habe, liefert folgende interessante Gleichungen

$$\int_0^\pi (\cos x+i\sin x\cos\varphi)^{n-1}\sin\varphi\,d\varphi = 2\sum_{n_1, n_2, \ldots, n_r} C_{n_1}^{\nu_1}(\cos x)\,C_{n_2}^{\nu_2}(\cos x)\ldots C_{n_r}^{\nu_r}(\cos x)$$

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[1]) „Über die Functionen $C_n^\nu(x)$", Sitzungsberichte der kais. Akad. d. Wissensch. 97. Band, Abth. IIa, S. 259—270.

$$\int_0^\infty J^{2n}(y)\sin\left(y\sin\frac{x}{2}\right)dy = 2\sin\frac{x}{2}\sum_{n_1,n_2,\ldots,n_r} C_{n_1}^{\nu_1}(\cos x)\,C_{n_2}^{\nu_2}(\cos x)\ldots C_{n_r}^{\nu_r}(\cos x)$$

$$\int_0^\infty J^{2n}(y)\sin\left(y\cos\frac{x}{2}\right)dy = (-1)^{n-1}\,2\cos\frac{x}{2}\sum_{n_1,n_2,\ldots,n_r} C_{n_1}^{\nu_1}(\cos x)\,C_{n_2}^{\nu_2}(\cos x)\ldots C_{n_r}^{\nu_r}(\cos x)$$

$$\int_0^1 C_{2n-2}^{2}\left(z\sin\frac{x}{2}\right)dz = (-1)^{n-1}\sum_{n_1,n_2,\ldots,n_r} C_{n_1}^{\nu_1}(\cos x)\,C_{n_2}^{\nu_2}(\cos x)\ldots C_{n_r}^{\nu_r}(\cos x)$$

$$\int_0^\infty J^{2n}(y)\sin(y\cos x)\,dy = (-1)^{n-1}\sum_{m_1,m_2,\ldots,m_r} C_{m_1}^{\nu_1}(\cos x)\,C_{m_2}^{\nu_2}(\cos x)\ldots C_{m_r}^{\nu_r}(\cos x)$$

$$\int_0^\infty J^{2n+1}(y)\cos(y\cos x)\,dy = (-1)^{n}\sum_{n_1',n_2',\ldots,n_r'} C_{n_1'}^{\nu_1}(\cos x)\,C_{n_2'}^{\nu_1}(\cos x)\ldots C_{n_r'}^{\nu_r}(\cos x)$$

$$\int_0^{\frac{\pi}{2}} \cos^n\varphi\,\left\{(\cos x\cos\varphi + i\sin\varphi)^{n-1} + (\cos x\cos\varphi - i\sin\varphi)^{n-1}\right\}d\varphi =$$

$$= \frac{\Pi(n-1)\pi}{2^{n-1}\Pi\left(n-\frac{1}{2}\right)}\sum_{n_1,n_2,\ldots,n_r} C_{n_1}^{\nu_1}(\cos x)\,C_{n_2}^{\nu_2}(\cos x)\ldots C_{n_r}^{\nu_r}(\cos x)$$

$$\int_0^{\frac{\pi}{2}} \sin^n\varphi\,\left\{(\cos x\sin\varphi + i\cos\varphi)^{n-1} + (\cos x\sin\varphi - i\cos\varphi)^{n-1}\right\}d\varphi =$$

$$= \frac{\Pi(n-1)\pi}{2^{n-1}\Pi\left(n-\frac{1}{2}\right)}\sum_{n_1,n_2,\ldots,n_r} C_{n_1}^{\nu_1}(\cos x)\,C_{n_2}^{\nu_2}(\cos x)\ldots C_{n_r}^{\nu_r}(\cos x),$$

$$\left(n_\lambda = 0,1,2,\ldots,n-1;\ m_\lambda = 0,1,2,\ldots,2n-1;\ n'_\lambda = 0,1,2,\ldots,2n;\ \sum_{\lambda=1}^{\lambda=r} n_\lambda = n-1;\ \sum_{\lambda=1}^{\lambda=r} m_\lambda = 2n-1;\ \sum_{\lambda=1}^{\lambda=r} n'_\lambda = 2n;\ \sum_{\lambda=1}^{\lambda=r}\nu_\lambda = 1\right)$$

$$\int_0^\pi\int_0^\pi\ldots\int_0^\pi \prod_{k=1}^{\frac{m-1}{2}}\left(\cos\frac{2k\pi}{m} + i\sin\frac{2k\pi}{m}\cos\varphi_k\right)^{n-1}\sin\varphi_k\,d\varphi_k = 2^{\frac{m-1}{2}}\left(\frac{n}{m}\right)$$

$$\int_0^1\int_0^1\ldots\int_0^1 \prod_{k=1}^{\frac{m-1}{2}} C_{2n-2}^{2}\left(z_k\sin\frac{k\pi}{m}\right)dz_k = (-1)^{\frac{(m-1)(n-1)}{2}}\left(\frac{n}{m}\right)$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\ldots\int_0^{\frac{\pi}{2}} \prod_{k=1}^{\frac{m-1}{2}} \cos^n\varphi_k\left\{\left(\cos\frac{2k\pi}{m}\cos\varphi_k + i\sin\varphi_k\right)^{n-1} + \left(\cos\frac{2k\pi}{m}\cos\varphi_k - i\sin\varphi_k\right)^{n-1}\right\}d\varphi_k =$$

$$= \frac{\pi^{\frac{m-1}{2}}\left(\Pi(n-1)\right)^{\frac{m-1}{2}}}{\left(2^{n-1}\Pi\left(n-\frac{1}{2}\right)\right)^{\frac{m-1}{2}}}\left(\frac{n}{m}\right)$$

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \sum_{\varepsilon} \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{2r-1} \cos\Big(y_1 \sin \frac{\pi}{4r+1} + \varepsilon_1 y_2 \sin \frac{2\pi}{4r+1} + \varepsilon_2 y_3 \frac{3\pi}{4r+1} + \ldots$$

$$+ \varepsilon_{2r-1} y_{2r} \sin \frac{2r\pi}{4r+1}\Big) dy_1\, dy_2 \ldots dy_{2r} \int_0^\pi \int_0^\pi \cdots \int_0^\pi J^0\big(\sqrt{y_1^2+y_2^2-2y_1y_2\cos\varphi_1}\big)\, J^0\big(\sqrt{y_3^2+y_4^2-2y_3y_4\cos\varphi_2}\big) \ldots$$

$$\ldots J^0\big(\sqrt{y_{2r-1}^2+y_{2r}^2-2y_{2r-1}y_{2r}\cos\varphi_r}\big) \cos 2n\varphi_1 \cos 2n\varphi_2 \ldots \cos 2n\varphi_r\, d\varphi_1\, d\varphi \ldots d\varphi_r =$$

$$= (-1)^r\, 2^{2r-1} \pi^r \Big(\frac{n}{4r+1}\Big) \sqrt{4r+1}$$

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \sum_{\varepsilon} \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{2r} \sin\Big(y_1 \sin \frac{\pi}{4r+3} + \varepsilon_1 y_2 \sin \frac{2\pi}{4r+3} + \ldots + \varepsilon_{2r} y_{2r+1} \sin \frac{(2r+1)\pi}{4r+3}\Big) dy_1\, dy_2 \ldots dy_{2r+1}\,.$$

$$.\int_0^\pi \int_0^\pi \cdots \int_0^\pi \prod_1^r{}_{\lambda}\, J^0\big(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\big) \cos 2n\varphi_\lambda\, J^{2n}(y_{2r+1})\, d\varphi_\lambda = (-1)^r\, 2^{2r} \pi^r \Big(\frac{n}{4r+3}\Big) \sqrt{4r+3}$$

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \sum_{\varepsilon} \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{2r-1} \cos\Big(y_1 \cos \frac{\pi}{4r+1} + \varepsilon_1 y_2 \cos \frac{2\pi}{4r+1} + \ldots + \varepsilon_{2r-1} y_{2r} \cos \frac{2r\pi}{4r+1}\Big) dy_1\, dy_2 \ldots dy_{2r}\,.$$

$$.\int_0^\pi \int_0^\pi \cdots \int_0^\pi \prod_1^r{}_{\lambda}\, J^0\big(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\big) \cos 2n\varphi_\lambda\, d\varphi_\lambda = (-1)^{r+1}\, 2^{2r-1} \Big(\frac{n}{4r+1}\Big) \pi^r$$

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \sum_{\varepsilon} \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{2r} \sin\Big(y_1 \cos \frac{\pi}{4r+3} + \varepsilon_1 y_2 \cos \frac{2\pi}{4r+3} + \ldots + \varepsilon_{2r} y_{2r+1} \cos \frac{(2r+1)\pi}{4r+1}\Big) dy_1\, dy_2 \ldots dy_{2r+1}\,.$$

$$\int_0^\pi \int_0^\pi \cdots \int_0^\pi \prod_1^r{}_{\lambda}\, J^0\big(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\big) \cos 2n\varphi_\lambda\, J^{2n}(y_{2r+1})\, d\varphi_\lambda = (-1)^{(n-1)(2r+1)+1+r}\, 2^{2r} \pi^r \Big(\frac{n}{4r+3}\Big)$$

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \int_0^\pi \int_0^\pi \cdots \int_0^\pi \sum_{\varepsilon} \cos\Big(y_1 \cos \frac{2\pi}{4r+1} + \varepsilon_1 y_2 \cos \frac{4\pi}{4r+1} + \varepsilon_2 y_3 \cos \frac{6\pi}{4r+1} + \ldots$$

$$+ \varepsilon_{2r-1} y_{2r} \cos \frac{4r\pi}{4r+1}\Big) \prod_1^r{}_{\lambda}\, J^0\big(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}2y_{2\lambda}\cos\varphi_\lambda}\big) \cos(2n+1)\varphi_\lambda\, dy_1\, dy_2 \ldots dy_{2r}\, d\varphi_\lambda = \frac{1}{2}\,(4\pi)^r \Big(\frac{2n+1}{4r+1}\Big)$$

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \int_0^\pi \int_0^\pi \cdots \int_0^\pi \sum_{\varepsilon} \cos\Big(y_1 \cos \frac{2\pi}{4r+3} + \varepsilon_1 y_2 \cos \frac{4\pi}{4r+3} + \varepsilon_2 y_3 \cos \frac{6\pi}{4r+3} + \ldots$$

$$+ \varepsilon_{2r} y_{2r+1} \cos \frac{(4r+2)\pi}{4r+3}\Big) \prod_1^r{}_{\lambda}\, J^0\big(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\big)\, J^{2n+1}(y_{2r+1}) \cos(2n+1)\varphi_\lambda\, dy_1\, dy_2 \ldots dy_{2r+1}\, d\varphi_\lambda =$$

$$= (-1)^n\, (4\pi)^r \Big(\frac{2n+1}{4r+3}\Big)$$

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \int_0^\pi \int_0^\pi \cdots \int_0^\pi \sum_{\varepsilon} \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{2r-1} \cos\Big(y_1 \cos \frac{2\pi}{4r+1} + \varepsilon_1 y_2 \cos \frac{4\pi}{4r+1} + \ldots + \varepsilon_{2r-1} y_{2r} \cos \frac{4r\pi}{4r+1}\Big).$$

$$.\prod_1^r{}_{\lambda}\, J^0\big(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\big) \cos 2n\varphi_\lambda\, dy_1\, dy_2 \ldots dy_{2r}\, d\varphi_\lambda = \frac{1}{2}\,(4\pi)^r \Big(\frac{n}{4r+1}\Big)$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon \varepsilon_1\varepsilon_2\ldots\varepsilon_{2r}\sin\Big(y_1\cos\frac{2\pi}{4r+3}+\varepsilon_1 y_2\cos\frac{4\pi}{4r+3}+\ldots$$

$$+\varepsilon_{2r}y_{2r+1}\cos\frac{(4r+2)\pi}{4r+3}\Big)J^{2n}(y_{2r+1})\prod_{\lambda=1}^{r}J^0\big(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\big)\cos 2n\varphi_\lambda\, dy_1\, dy_2\ldots dy_{2r+1}\,d\varphi_r=$$

$$=(-1)^n(4\pi)^r\binom{n}{4r+3}$$

wo die Grössen ε_λ die Werthe $+1$ und -1 besitzen und die Summationen über alle Combinationen der Grössen ε_λ auszudehnen sind.

Von den speciellen Fällen dieser Gleichungen mögen die folgenden besonders angeführt werden:

$$\int_0^\pi\int_0^\pi\cdots\int_0^\pi\prod_{k=1}^{\frac{m-1}{2}}\Big(\cos\frac{2k\pi}{m}+i\sin\frac{2k\pi}{m}\cos\varphi_k\Big)^{2^\lambda 3^\mu 5^\nu-1}\sin\varphi_k d\varphi_k=(-1)^{\lambda\left[\frac{m+1}{4}\right]+\mu\left[\frac{m+1}{6}\right]+\nu\left[\frac{m+2}{5}\right]}2^{m-1}$$

$$\int_0^1\int_0^1\cdots\int_0^1\prod_{k=1}^{\frac{m-1}{2}}C^2_{2^{\lambda+1}3^\mu 5^\nu-2}\Big(z_k\sin\frac{k\pi}{m}\Big)dz_k=(-1)^{\frac{(m-1)(2^\lambda 3^\mu 5^\nu-1)}{2}+\lambda\left[\frac{m+1}{4}\right]+\mu\left[\frac{m+1}{6}\right]+\nu\left[\frac{m+2}{5}\right]}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}\prod_{k=1}^{\frac{m-1}{2}}\cos^{2^\lambda 3^\mu 5^\nu}\varphi_k\Big\{\Big(\cos\frac{2k\pi}{m}\cos\varphi_k+i\sin\varphi_k\Big)^{2^\lambda 3^\mu 5^\nu-1}+\Big(\cos\frac{2k\pi}{m}\cos\varphi_k-i\sin\varphi_k\Big)^{2^\lambda 3^\mu 5^\nu-1}\Big\}d\varphi_k=$$

$$=\frac{\pi^{\frac{m-1}{2}}\big(\Pi(2^\lambda 3^\mu 5^\nu-1)\big)^{\frac{m-1}{2}}}{\Big(2^{2^\lambda 3^\mu 5^\nu-1}\,\Pi\Big(2^\lambda 3^\mu 5^\nu-\frac{1}{2}\Big)\Big)^{\frac{m-1}{2}}}(-1)^{\lambda\left[\frac{m+1}{4}\right]+\mu\left[\frac{m+1}{6}\right]+\nu\left[\frac{m+2}{5}\right]}$$

$$\int_0^\pi\int_0^\pi\cdots\int_0^\pi\prod_{k=1}^{7n+\frac{r-1}{2}}\Big(\cos\frac{2k\pi}{14n+r}+i\sin\frac{2k\pi}{14n+r}\cos\varphi_k\Big)^6\sin\varphi_k d\varphi_k=(-1)^{n-\zeta}2^{7n+\frac{r-1}{2}}$$

$$\int_0^1\int_0^1\cdots\int_0^1\prod_{k=1}^{7n+\frac{r-1}{2}}C_{12}^2\Big(z_k\sin\frac{k\pi}{14n+r}\Big)dz_k=(-1)^{n+\zeta}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}\prod_{k=1}^{7n+\frac{r-1}{2}}\cos^7\varphi_k\Big\{\Big(\cos\frac{2k\pi}{m}\cos\varphi_k+i\sin\varphi_k\Big)^6+\Big(\cos\frac{2k\pi}{m}\cos\varphi_k-i\sin\varphi_k\Big)^6\Big\}d\varphi_k=$$

$$=(-1)^{n+\zeta}\left[\frac{45\pi}{4\,\Pi\left(\frac{13}{2}\right)}\right]^{7n+\frac{r-1}{2}}$$

$$\int_0^\pi\int_0^\pi\cdots\int_0^\pi\prod_{k=1}^{11n+\frac{r-1}{2}}\Big(\cos\frac{2k\pi}{22n+r}+i\sin\frac{2k\pi}{22n+r}\cos\varphi_k\Big)^{10}\sin\varphi_k d\varphi_k=(-1)^{n+\eta}2^{11n+\frac{r-1}{2}}$$

$$\int_0^1\int_0^1\cdots\int_0^1\prod_{k=1}^{11n+\frac{r-1}{2}}C_{20}^2\Big(z_k\sin\frac{k\pi}{22n+r}\Big)dz_k=(-1)^{n+\eta}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}\prod_{1}^{11n+\frac{r-1}{2}}\boxed{k}\cos^{11}\varphi_k\left\{\left(\cos\frac{2k\pi}{22n+r}\cos\varphi_k+i\sin\varphi_k\right)^{10}+\left(\cos\frac{2k\pi}{22n+r}-i\sin\varphi_k\right)^{10}\right\}d\varphi_k=$$

$$=(-1)^{n+\eta}\left[\frac{14175\pi}{4\Pi\left(\frac{21}{2}\right)}\right]^{11n+\frac{r-1}{2}}$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\varepsilon_1\varepsilon_2\ldots\varepsilon_{2r-1}\cos\left(y_1\sin\frac{\pi}{4r+1}+\varepsilon_1 y_2\sin\frac{2\pi}{4r+1}+\ldots+\varepsilon_{2r-1}y_{2r}\sin\frac{2r\pi}{4r+1}\right)\cdot$$

$$\cdot\prod_1^r\boxed{\lambda}J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)\cos\left(2^{\lambda_1+1}3^\mu5^\nu\varphi_\lambda\right)dy_1dy_2\ldots dy_{2r}d\varphi_\lambda=$$

$$=(-1)^{r+\lambda_1r+\mu\left[\frac{2r+1}{3}\right]+\nu\left[\frac{4r+3}{5}\right]}(4\pi)^r\frac{\sqrt{4r+1}}{2}$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\varepsilon_1\varepsilon_2\ldots\varepsilon_{2r}\sin\left(y_1\sin\frac{\pi}{4r+3}+\varepsilon_1y_2\sin\frac{2\pi}{4r+3}+\ldots+\varepsilon_{2r}y_{2r+1}\sin\frac{(2r+1)\pi}{4r+3}\right)\cdot$$

$$\cdot\prod_1^r\boxed{\lambda}J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)J^{2^{\lambda_1+1}3^\mu5^\nu}(y_{2r+1})\cos\left(2^{\lambda+1}3^\mu5^\nu\varphi_\lambda\right)dy_1dy_2\ldots dy_{2r+1}d\varphi_\lambda=$$

$$=(-1)^{r+\lambda_1(r+1)+\mu\left[\frac{2r+2}{3}\right]+\nu\left[\frac{4r+5}{5}\right]}(4\pi)^r\sqrt{4r+3}$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\varepsilon_1\varepsilon_2\ldots\varepsilon_{2r-1}\cos\left(y_1\cos\frac{\pi}{4r+1}+\varepsilon_1y_2\cos\frac{2\pi}{4r+1}+\ldots+\varepsilon_{2r-1}y_{2r}\cos\frac{2r\pi}{4r+1}\right)\cdot$$

$$\cdot\prod_1^r\boxed{\lambda}J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)\cos\left(2^{\lambda_1+1}3^\mu5^\nu\varphi_\lambda\right)dy_1dy_2\ldots dy_{2r}d\varphi_\lambda=$$

$$=(-1)^{r+1+\lambda_1r+\mu\left[\frac{2r+1}{3}\right]+\nu\left[\frac{4r+3}{5}\right]}2^{2r-1}\pi^r$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\varepsilon_1\varepsilon_2\ldots\varepsilon_{2r}\sin\left(y_1\cos\frac{\pi}{4r+3}+\varepsilon_1y_2\cos\frac{2\pi}{4r+3}+\ldots\varepsilon_{2r-1}y_{2r}\cos\frac{(2r+1)\pi}{4r+3}\right)\cdot$$

$$\cdot\prod_1^r\boxed{\lambda}J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)J^{2^{\lambda_1+1}3^\mu5^\nu}(y_{2r+1})\cos\left(2^{\lambda_1+1}3^\mu5^\nu\varphi_\lambda\right)dy_1dy_2\ldots dy_{2r+1}d\varphi_\lambda=$$

$$=(-1)^{2^{\lambda_1}+\lambda_1+r(\lambda_1+1)+\mu\left[\frac{2r+2}{3}\right]+\nu\left[\frac{4r+5}{5}\right]}(4\pi)^r$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\cos\left(y_1\cos\frac{2\pi}{4r+1}+\varepsilon_1y_2\cos\frac{4\pi}{4r+1}+\ldots+\varepsilon_{2r-1}y_{2r}\cos\frac{4r\pi}{4r+1}\right)\cdot$$

$$\cdot\prod_1^r\boxed{\lambda}J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)\cos\left(3^\mu5^\nu\varphi_\lambda\right)dy_1\,dy_2\ldots dy_{2r}d\varphi_\lambda=(-1)^{\mu\left[\frac{2r+1}{3}\right]+\nu\left[\frac{4r+3}{5}\right]}2^{2r-1}\pi^r$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\cos\left(y_1\cos\frac{2\pi}{4r+3}+\varepsilon_1y_2\cos\frac{4\pi}{4r+3}+\ldots+\varepsilon_{2r}y_{2r+1}\cos\frac{(4r+2)\pi}{4r+3}\right)\cdot$$

$$\cdot\prod_1^r\boxed{\lambda}J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)J^{3^\mu5^\nu+1}(y_{2r+1})\cos\left(3^\mu5^\nu\varphi_\lambda\right)dy_1\,dy_2\ldots dy_{2r+1}d\varphi_\lambda=$$

$$=(-1)^{3^\mu5^\nu+\mu\left[\frac{2r+2}{3}\right]+\nu\left[\frac{4r+5}{5}\right]}(4\pi)^r$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\cos\left(y_1\cos\frac{2\pi}{4r+1}+\varepsilon_1 y_2\cos\frac{4\pi}{4r+1}+\ldots+\varepsilon_{2r-1}y_{2r}\cos\frac{4r\pi}{4r+1}\right)\cdot$$

$$\cdot\prod_1^r{}_\lambda J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)\cos\left(2^{\lambda_1+1}3^\mu5^\nu\varphi_\lambda\right)dy_1dy_2\ldots dy_{2r}d\varphi_\lambda=(-1)^{\mu\left[\frac{2r+1}{3}\right]+\nu\left[\frac{4r+3}{5}\right]+r(\lambda_1+1)}2^{2r-1}\pi^r$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\sin\left(y_1\cos\frac{2\pi}{4r+3}+\varepsilon_1 y_3\cos\frac{4\pi}{4r+3}+\ldots+\varepsilon_{2r}y_{2r+1}\cos\frac{(4r+2)\pi}{4r+1}\right)\cdot$$

$$\cdot\prod_1^r{}_\lambda J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)J^{2^{\lambda_1+1}3^\mu5^\nu}(y_{2r+1})\cos\left(2^{\lambda_1+1}3^\mu5^\nu\varphi_\lambda\right)dy_1\,dy_2\ldots dy_{2r+1}d\varphi_\lambda=$$

$$=(-1)^{2^{\lambda_1}+(r+1)(\lambda_1+1)+\mu\left[\frac{2r+2}{3}\right]+\nu\left[\frac{4r+5}{5}\right]}(4\pi)^r$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\varepsilon_1\varepsilon_2\ldots\varepsilon_{2r-1}\cos\left(y_1\sin\frac{\pi}{4r+1}+\varepsilon_1y_2\sin\frac{2\pi}{4r+1}+\ldots+\varepsilon_{2r-1}y_{2r}\sin\frac{2r\pi}{4r+1}\right)\cdot$$

$$\cdot\prod_1^r{}_\lambda J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)\cos12\varphi_\lambda dy_1\,dy_2\ldots dy_{2r}d\varphi_\lambda=(-1)^{r+\left[\frac{2r+3}{6}\right]}2^{2r-1}\pi^r\sqrt{4r+1}$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\varepsilon_1\varepsilon_2\ldots\varepsilon_{2r}\sin\left(y_1\sin\frac{\pi}{4r+3}+\varepsilon_1y_2\sin\frac{2\pi}{4r+3}+\ldots\right.$$

$$\left.+\varepsilon_{2r}y_{2r}\sin\frac{(2r+1)\pi}{4r+3}\right)J^{12}(y_{2r+1})\prod_1^r{}_\lambda J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)\cos12\varphi_\lambda dy_1dy_2\ldots dy_{2r+1}d\varphi_\lambda=$$

$$=(-1)^{r+\left[\frac{r+2}{3}\right]}(4\pi)^r\sqrt{4r+3}$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\varepsilon_1\varepsilon_2\ldots\varepsilon_{2r-1}\cos\left(y_1\cos\frac{\pi}{4r+1}+\varepsilon_1y_2\cos\frac{2\pi}{4r+1}+\ldots+\varepsilon_{2r-1}y_{2r}\cos\frac{2r\pi}{4r+1}\right)\cdot$$

$$\cdot\prod_1^r{}_\lambda J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)\cos12\varphi_\lambda dy_1dy_2\ldots dy_{2r}d\varphi_\lambda=(-1)^{r+1+\left[\frac{2r+3}{6}\right]}2^{2r-1}\pi^r$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\varepsilon_1\varepsilon_2\ldots\varepsilon_{2r}\sin\left(y_1\cos\frac{\pi}{4r+1}+\varepsilon_1y_2\cos\frac{2\pi}{4r+1}+\ldots\right.$$

$$\left.+\varepsilon_{2r}y_{2r+1}\cos\frac{(2r+1)\pi}{4r+1}\right)J^{12}(y_{2r+1})\prod_1^r{}_\lambda J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)\cos12\varphi_\lambda dy_1\,dy_2\ldots dy_{2r+1}d\varphi_\lambda=$$

$$=(-1)^{r+\left[\frac{r+2}{3}\right]}2^{2r}\pi^r$$

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\int_0^\pi\int_0^\pi\cdots\int_0^\pi\sum_\varepsilon\varepsilon_1\varepsilon_2\ldots\varepsilon_{2r-1}\cos\left(y_1\cos\frac{2\pi}{4r+1}+\varepsilon_1y_2\cos\frac{4\pi}{4r+1}+\ldots+\varepsilon_{2r-1}y_{2r}\cos\frac{4r\pi}{4r+1}\right)\cdot$$

$$\cdot\prod_0^r{}_\lambda J^0\left(\sqrt{y_{2\lambda-1}^2+y_{2\lambda}^2-2y_{2\lambda-1}y_{2\lambda}\cos\varphi_\lambda}\right)\cos12\varphi_\lambda dy_1dy_2\ldots dy_{2r}d\varphi_\lambda=(-1)^{\left[\frac{2r+3}{6}\right]}2^{2r-1}\pi^r$$

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \int_0^\pi \int_0^\pi \cdots \int_0^\pi \sum_\varepsilon \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{2r} \sin\left(y_1 \cos\frac{2\pi}{4r+3} + \varepsilon_1 y_2 \cos\frac{4\pi}{4r+3} + \ldots\right.$$

$$\left. + \varepsilon_{2r} y_{2r+1} \cos\frac{(4r+2)\pi}{4r+3}\right) J^{12}(y_{2r+1}) \prod_1^r {}_{\lambda}\, J^0\left(\sqrt{y_{2\lambda-1}^2 + y_{2\lambda}^2 - 2y_{2\lambda-1} y_{2\lambda} \cos\varphi_\lambda}\right) \cos 12\varphi_\lambda \, dy_1 \, dy_2 \ldots dy_{2r+1} \, d\varphi_\lambda =$$

$$= (-1)^{\left[\frac{r+2}{3}\right]} (4\pi)^r$$

$$\int_0^\pi \int_0^\pi \cdots \int_0^\pi \prod_1^{\frac{m-1}{2}} {}_k \left(\cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{m} \cos\varphi_k\right)^5 \sin\varphi_k \, d\varphi_k = (-1)^{\left[\frac{m+5}{12}\right]} 2^{m-1}$$

$$\int_0^1 \int_0^1 \cdots \int_0^1 \prod_1^{\frac{m-1}{2}} {}_k \, C_{10}^2\left(z_k \sin\frac{k\pi}{m}\right) dz_k = (-1)^{\frac{m-1}{2} + \left[\frac{m+5}{12}\right]}$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \prod_1^{\frac{m-1}{2}} {}_k \cos^6\varphi_k \left\{ \left(\cos\frac{2k\pi}{m} \cos\varphi_k + i \sin\varphi_k\right)^5 + \left(\cos\frac{2k\pi}{m} \cos\varphi_k - i \sin\varphi_k\right)^5 \right\} d\varphi_k =$$

$$= (-1)^{\left[\frac{m+5}{12}\right]} \left(\frac{2^4 \sqrt{\pi}}{693}\right)^{\frac{m-1}{2}}$$

($\zeta = 0$ für $r = 1, 3, 9$, $\zeta = 1$ für $r = 5, 11, 13$; $\eta = 0$ für $r = 1, 5, 7, 9, 19$, $\eta = 1$ für $r = 3, 13, 15, 17, 21$).

Multiplicirt man die Gleichung 14) mit $\dfrac{\varphi_1^{(n)}(y)}{2^n \Pi\left(n+\nu-\frac{1}{2}\right)}$ und summirt bezüglich n von 0 bis ∞, so erhält man die Relation:

$$\sum_{n=0}^{n=\infty} \frac{\varphi_1^{(n)}(y)\, C_n^\nu(x)}{2^n \Pi\left(n+\nu-\frac{1}{2}\right)} = \frac{1}{\Pi(\nu-1)\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \cos^{2\nu-1}\varphi \, d\varphi \sum_{n=0}^{n=\infty} \frac{\varphi_1^{(n)}(y)}{\Pi(n)} \{(x\cos\varphi + i\sin\varphi)^n + (x\cos\varphi - i\sin\varphi)^n\} \cos^n\varphi$$

$$= \frac{1}{\Pi(\nu-1)\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \, d\varphi \sum_{n=0}^{n=\infty} \frac{\varphi_1^{(n)}(y)}{\Pi(n)} \{(x\sin\varphi + i\cos\varphi)^n + (x\sin\varphi - i\cos\varphi)^n\} \sin^n\varphi$$

oder

15)
$$\sum_{n=0}^{n=\infty} \frac{\varphi_1^{(n)}(y)\, C_n^\nu(x)}{2^n \Pi\left(n+\nu-\frac{1}{2}\right)} = \frac{1}{\Pi(\nu-1)\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \cos^{2\nu-1}\varphi \left\{ \varphi_1\left(y + x\cos^2\varphi + \frac{i\sin 2\varphi}{2}\right) + \right.$$

$$\left. + \varphi_1\left(y + x\cos^2\varphi - \frac{i\sin 2\varphi}{2}\right) \right\} d\varphi$$

$$= \frac{1}{\Pi(\nu-1)\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \left\{ \varphi_1\left(y + x\sin^2\varphi + \frac{i\sin 2\varphi}{2}\right) + \right.$$

$$\left. + \varphi_1\left(y + x\sin^2\varphi - \frac{i\sin 2\varphi}{2}\right) \right\} d\varphi$$

aus welcher die Formel

$$16)\quad \varphi_1^{(n)}(y) = \frac{2^{n-1}(n+\nu)\Pi(n)\Pi\left(n+\nu-\frac{1}{2}\right)}{\Pi(n+2\nu-1)\Pi(\nu-1)\sqrt{\pi}}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}}C_n^\nu(x).$$

$$.\cos^{2\nu-1}\varphi\left\{\varphi_1\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+\varphi_1\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}dx\,d\varphi$$

$$= \frac{2^{n-1}(n+\nu)\Pi(n)\Pi\left(n+\nu-\frac{1}{2}\right)}{\Pi(n+2\nu-1)\Pi(\nu-1)\sqrt{\pi}}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}}C_n^\nu(x).$$

$$.\sin^{2\nu-1}\varphi\left\{\varphi_1\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)+\varphi_1\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}dx\,d\varphi$$

folgt, welche eine interessante Darstellung der n^{ten} Ableitung einer Function durch ein bestimmtes Doppelintegral liefert.

Auf demselben Wege findet man die entsprechenden Formeln:

$$\sum_{n=0}^{n=\infty}\frac{\varphi^{(n)}(y_1)\,C_{2n}^\nu(x)}{\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)} = \frac{1}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{\varphi_1\left(y+x^2\cos^4\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\right.$$

$$\left.+\varphi_1\left(y+x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}d\varphi$$

$$= \frac{1}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{\varphi_1\left(y+x^2\sin^4\varphi+xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\right.$$

$$\left.+\varphi_1\left(y+x^2\sin^4\varphi-xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=\infty}\frac{\varphi^{(n)}(y_1)\,C_{2n+1}^\nu(x)}{\Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)} = \frac{1}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{(x\cos\varphi+\right.$$

$$+i\sin\varphi)\,\varphi_1\left(y+x^2\cos^4\varphi+x\,i\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+(x\cos\varphi-i\sin\varphi)\,\varphi_1\left(y+x^2\cos^4\varphi+\right.$$

$$\left.\left.+x\,i\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}d\varphi$$

$$= \frac{1}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{(x\sin\varphi+i\cos\varphi)\,\varphi_1\left(y+x^2\sin^4\varphi+x\,i\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\right.$$

$$\left.+(x\sin\varphi-i\cos\varphi)\,\varphi_1\left(y+x^2\sin^4\varphi-xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}d\varphi$$

$$\varphi_1^{(n)}(y) = \frac{2^{2n-1}(2n+\nu)\Pi(2n)\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)}{\Pi(2n+2\nu-1)\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}}C_{2n}^\nu(x).$$

$$\cos^{2\nu-1}\varphi\left\{\varphi_1\left(y+x^2\cos^4\varphi+x\,i\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\varphi_1\left(y+x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}dx\,d\varphi$$

$$=\frac{2^{2n-1}(2n+\nu)\Pi(2n)\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)}{\Pi(2n+2\nu-1)\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}}C_{2n}^\nu(x)\,.$$

$$\cdot\sin^{2\nu-1}\varphi\left\{\varphi_1\left(y+x^2\sin^4\varphi+xi\sin^2\varphi\sin2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\varphi_1\left(y+x^2\sin^4\varphi-xi\sin^2\varphi\sin2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}dx\,d\varphi$$

$$\varphi_1^{(n)}(y)=\frac{2^{2n}(2n+\nu+1)\Pi(2n+1)\Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)}{\Pi(2n+2\nu)\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}}C_{2n+1}^\nu(x)$$

$$\cdot\cos^{2\nu-1}\varphi\left\{(x\sin\varphi+i\cos\varphi)\,\varphi_1\left(y+x^2\cos^4\varphi+xi\cos^2\varphi\sin2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\right.$$

$$\left.+(x\cos\varphi-i\sin\varphi)\,\varphi_1\left(y+x^2\cos^4\varphi-xi\sin2\varphi\cos^2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}dx\,d\varphi$$

$$=\frac{2^{2n}(2n+\nu+1)\Pi(2n+1)\Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)}{\Pi(2n+2\nu)\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}}C_{2n+1}^\nu(x)\,.$$

$$\cdot\sin^{2\nu-1}\varphi\left\{(x\sin\varphi+i\cos\varphi)\,\varphi_1\left(y+x^2\sin^4\varphi+xi\sin^2\varphi\sin2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\right.$$

$$\left.+(x\sin\varphi-i\cos\varphi)\,\varphi_1\left(y+x^2\sin^4\varphi-xi\sin^2\varphi\sin2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}dx\,d\varphi$$

Multiplicirt man ferner die Gleichungen

$$z^n=\frac{(n+\nu)\Pi(n)}{2^{2\nu-1}\Pi(n+2\nu-1)}\left[\frac{\Pi(2\nu-1)}{\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}C_n^\nu(x)\,dx}{(1-2zx+z^2)^\nu}$$

$$\frac{z^n}{1-z^2}=\frac{\Pi(n)\nu}{2^{2\nu-2}\Pi(n+2\nu-1)}\left[\frac{\Pi(2\nu-1)}{\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}C_n^\nu(x)\,dx}{(1-2zx+z^2)^{\nu+1}}$$

$$c^nF\left(\frac{\alpha+n}{2},\frac{\alpha+n+1}{2},\frac{\beta+n}{2},\frac{\beta+n+1}{2},n+\nu+1,\frac{\gamma+n}{2},\frac{\gamma+n+1}{2},c^2\right)=$$

$$=\frac{2^{n+1}\Pi(n)\Pi(n+\nu)\Pi(\alpha-1)\Pi(\beta-1)\Pi(\gamma+n-1)}{\Pi(\nu-1)\Pi(\alpha+n-1)\Pi(\beta+n-1)\Pi(\gamma-1)\Pi(n+2\nu-1)}\int_{-1}^{+1}F(\alpha,\beta,\gamma,cx)(1-x^2)^{\frac{2\nu-1}{2}}C_n^\nu(x)\,dx$$

und 7*a* bez. mit

$$\frac{\Pi(n+2\nu-1)\varphi_1^{(n)}(y)}{2^n(n+\nu)\Pi(n)\Pi\left(n+\nu-\frac{1}{2}\right)},\quad\frac{\Pi(n+2\nu-1)\varphi_1^{(n)}(y)}{2^n\Pi\left(n+\nu-\frac{1}{2}\right)\Pi(n)},\quad\frac{\Pi(\alpha+n-1)\Pi(\beta+n-1)\Pi(n+2\nu-1)\varphi_1^{(n)}(y)}{2^{2n+1}\Pi(n+\nu)\Pi(\gamma+n-1)\Pi(n)\Pi\left(n+\nu-\frac{1}{2}\right)}$$

und summirt bezüglich n von 0 bis ∞, so ergeben sich unter Berücksichtigung der eben aufgestellten Gleichungen die Formeln:

$$\sum_{n=0}^{n=\infty}\frac{\Pi(n+2\nu-1)\varphi_1^{(n)}(y)z^n}{2^n\Pi(n)\Pi\left(n+\nu-\frac{1}{2}\right)(n+\nu)}=\frac{2}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}\frac{(1-x^2)^{\frac{2\nu-1}{2}}}{(1-2zx+z^2)^\nu}.$$

$$.\cos^{2\nu-1}\varphi\left\{\varphi_1\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+\varphi_1\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}dx\,d\varphi$$

$$=\frac{2}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}\frac{(1-x^2)^{\frac{2\nu-1}{2}}}{(1-2zx+z^2)^\nu}.$$

$$.\sin^{2\nu-1}\varphi\left\{\varphi_1\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)+\varphi_1\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}dx\,d\varphi$$

$$\sum_{n=0}^{n=\infty}\frac{\Pi(n+2\nu-1)\varphi_1^{(n)}(y)z^n}{2^n\Pi(n)\Pi\left(n+\nu-\frac{1}{2}\right)}=\frac{2\nu}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-2)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2(1-z^2)\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}\frac{(1-x^2)^{\frac{2\nu-1}{2}}}{(1-2zx+z^2)^{\nu+1}}.$$

$$\cos^{2\nu-1}\varphi\left\{\varphi_1\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+\varphi_1\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}dx\,d\varphi$$

$$=\frac{2\nu}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2(1-z^2)\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}\frac{(1-x^2)^{\frac{2\nu-1}{2}}}{(1-2zx+z^2)^{\nu+1}}.$$

$$.\sin^{2\nu-1}\varphi\left\{\varphi_1\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)+\varphi_1\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}dx\,d\varphi$$

$$\sum_{n=0}^{n=\infty}\frac{\Pi(\alpha+n-1)\Pi(\beta+n-1)\Pi(n+2\nu-1)\varphi_1^{(n)}(y)c^nF\left(\frac{\alpha+n}{2},\frac{\alpha+n+1}{2},\frac{\beta+n}{2},\frac{\beta+n+1}{2},n+\nu+1,\frac{\gamma+n+1}{2},\frac{\gamma+n}{2},c^2\right)}{2^{2n+1}\Pi(n)\Pi(n+\nu)\Pi(\gamma+n-1)\Pi\left(n+\nu-\frac{1}{2}\right)}=$$

$$=\frac{\Pi(\alpha-1)\Pi(\beta-1)}{\sqrt{\pi}\,\Pi(\gamma-1)[\Pi(\nu-1)]^2}\int_{-1}^{+1}F(\alpha,\beta,\gamma,cx)(1-x^2)^{\frac{2\nu-1}{2}}dx\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{\varphi_1\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+\right.$$

$$\left.+\varphi_1\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}d\varphi$$

$$=\frac{\Pi(\alpha-1)\Pi(\beta-1)}{\sqrt{\pi}\,\Pi(\gamma-1)[\Pi(\nu-1)]^2}\int_{-1}^{+1}F(\alpha,\beta,\gamma,cx)(1-x^2)^{\frac{2\nu-1}{2}}dx\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{\varphi_1\left(y+x\sin^2\varphi+\frac{\sin 2\varphi}{2}\right)+\right.$$

$$\left.+\varphi_1\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=\infty}(-1)^n\frac{\varphi_1^{(n)}(y)\,C_n^{\nu}(\cos x)}{\Pi\left(\frac{2n-1}{2}\right)\Pi\left(2n+2\nu-\frac{1}{2}\right)}=-\frac{\Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(\nu-1)\,\Pi(2\nu-1)\sqrt{\pi}\sin^{2\nu-1}\frac{x}{2}}\int_0^x\int_0^{\frac{\pi}{2}}(\cos\varphi-\cos x)^{\nu-1}\cos\frac{\varphi}{2}\,.$$

$$.\cos^{4\nu-1}\psi\left\{\varphi_1\left(y+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)+\right.$$

$$\left.+\varphi_1\left(y+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)\right\}d\varphi\,d\psi$$

$$=\frac{\Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(\nu-1)\,\Pi(2\nu-1)\sqrt{\pi}\sin^{2\nu-1}\frac{x}{2}}\int_0^x\int_0^{\frac{\pi}{2}}(\cos\varphi-\cos x)^{\nu-1}\cos\frac{\varphi}{2}\,.$$

$$.\sin^{4\nu-1}\psi\left\{\varphi_1\left(y+\sin^2\frac{\varphi}{2}\sin^4\psi+i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)+\right.$$

$$\left.+\varphi_1\left(y+\sin^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)\right\}d\varphi\,d\psi$$

$$\sum_{n=0}^{n=\infty}\frac{\varphi_1^{(n)}(y)\,C_n^{\nu}(\cos x)}{\Pi\left(\frac{2n-1}{2}\right)\Pi\left(2n+2\nu-\frac{1}{2}\right)}=\frac{\Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(\nu-1)\,\Pi(2\nu-1)\sqrt{\pi}\cos^{2\nu-1}\frac{x}{2}}\int_x^{\pi}\int_0^{\frac{\pi}{2}}(\cos x-\cos\varphi)^{\nu-1}\sin\frac{\varphi}{2}\,.$$

$$.\cos^{4\nu-1}\psi\left\{\varphi_1\left(y+\cos^2\frac{\varphi}{2}\cos^4\psi+i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)+\right.$$

$$\left.+\varphi_1\left(y+\cos^2\frac{\varphi}{2}\cos^4\psi-i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)\right\}d\varphi\,d\psi$$

$$=\frac{\Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(\nu-1)\,\Pi(2\nu-1)\sqrt{\pi}\cos^{2\nu-1}\frac{x}{2}}\int_x^{\pi}\int_0^{\frac{\pi}{2}}(\cos x-\cos\varphi)^{\nu-1}\sin\frac{\varphi}{2}\,.$$

$$.\sin^{4\nu-1}\psi\left\{\varphi_1\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi+i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)+\right.$$

$$\left.+\varphi_1\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi-i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)\right\}d\varphi\,d\psi$$

$$\sum_{n=0}^{n=\infty}(-1)^n\frac{\varphi_1^{(n)}(y)\,C_n^{\nu}(\cos x)}{\Pi\left(\frac{2n+1}{2}\right)\Pi\left(2n+2\nu+\frac{1}{2}\right)}=\frac{\Pi\left(\frac{2\nu-3}{2}\right)}{\Pi(\nu-2)\,\Pi(2\nu-2)\sqrt{\pi}\sin^{2\nu-1}\frac{x}{2}}\int_0^x\int_0^{\frac{\pi}{2}}(\cos\varphi-\cos x)^{\nu-2}\sin\varphi\,.$$

$$.\cos^{4\nu-3}\psi\left\{\left(\sin\frac{\varphi}{2}\cos\psi+i\sin\psi\right)\varphi_1\left(y+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)+\right.$$

$$\left.+\left(\sin\frac{\varphi}{2}\cos\psi-i\sin\psi\right)\varphi_1\left(y+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)\right\}d\varphi\,d\psi$$

$$= \frac{\Pi\left(\frac{2\nu-3}{2}\right)}{\Pi(\nu-2)\,\Pi(2\nu-2)\sqrt{\pi}\,\sin^{2\nu-1}\frac{x}{2}} \int_0^x \int_0^{\frac{\pi}{2}} (\cos\varphi - \cos x)^{\nu-2} \sin\varphi\,.$$

$$.\sin^{4\nu-3}\psi \left\{ \left(\sin\frac{\varphi}{2}\sin\psi + i\cos\psi\right) \varphi_1\left(y + \sin^2\frac{\varphi}{2}\sin^4\psi + i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi - \frac{\sin^2 2\psi}{4}\right) + \right.$$

$$\left. + \left(\sin\frac{\varphi}{2}\sin\psi - i\cos\psi\right) \varphi_1\left(y + \sin^2\frac{\varphi}{2}\sin^4\psi - i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi - \frac{\sin^2 2\psi}{4}\right)\right\} d\varphi\, d\psi$$

$$\sum_{n=0}^{n=\infty} \frac{\varphi_1^{(n)}(y)\, C_n^\nu(\cos x)}{\Pi\left(\frac{2n+1}{2}\right)\Pi\left(2n+2\nu+\frac{1}{2}\right)} = \frac{\Pi\left(\frac{2\nu-3}{2}\right)}{\Pi(\nu-2)\,\Pi(2\nu-2)\sqrt{\pi}\,\cos^{2\nu-1}\frac{x}{2}} \int_x \int_0^{\frac{\pi}{2}} (\cos x - \cos\varphi)^{\nu-2} \sin\varphi\,.$$

$$.\cos^{4\nu-3}\psi \left\{ \left(\cos\frac{\varphi}{2}\cos\psi + i\sin\psi\right) \varphi_1\left(y + \cos^2\frac{\varphi}{2}\cos^4\psi + i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi - \frac{\sin^2 2\psi}{4}\right) + \right.$$

$$\left. + \left(\cos\frac{\varphi}{2}\cos\psi - i\sin\psi\right) \varphi_1\left(y + \cos^2\frac{\varphi}{2}\cos^4\psi - i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi - \frac{\sin^2 2\psi}{4}\right)\right\} d\varphi\, d\psi$$

$$= \frac{\Pi\left(\frac{2\nu-3}{2}\right)}{\Pi(\nu-2)\,\Pi(2\nu-2)\sqrt{\pi}\,\cos^{2\nu-1}\frac{x}{2}} \int_x^\pi \int_0^{\frac{\pi}{2}} (\cos x - \cos\varphi)^{\nu-2} \sin\varphi\,.$$

$$.\sin^{4\nu-3}\psi \left\{ \left(\cos\frac{\varphi}{2}\sin\psi + i\cos\psi\right) \varphi_1\left(y + \cos^2\frac{\varphi}{2}\sin^4\psi + .\, i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi + \frac{\sin^2 2\psi}{4}\right) + \right.$$

$$\left. + \left(\cos\frac{\varphi}{2}\sin\psi - i\cos\psi\right) \varphi_1\left(y + \cos^2\frac{\varphi}{2}\sin^4\psi - i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi - \frac{\sin^2 2\psi}{4}\right)\right\} d\varphi\, d\psi$$

Setzt man in den eben abgeleiteten Gleichungen speciell der Reihe nach

$$\varphi_1(y) = y^{-\frac{\mu}{2}} J^\mu(\sqrt{y})$$

$$\varphi_1(y) = y^{\frac{\mu}{2}} J^\mu(\sqrt{y})$$

$$\varphi_1(y) = C_r^\mu(y)$$

und berücksichtigt, dass

$$\left[y^{-\frac{\mu}{2}} J^\mu(\sqrt{y})\right]^{(n)} = \left(-\frac{1}{2}\right)^n y^{-\frac{\mu+n}{2}} J^{\frac{\mu+n}{2}}(\sqrt{y})$$

$$\left[y^{\frac{\mu}{2}} J^\mu(\sqrt{y})\right]^{(n)} = \left(\frac{1}{2}\right)^n y^{\frac{\mu-n}{2}} J^{\frac{\mu-n}{2}}(\sqrt{y})$$

$$\left[C_r^\mu(x)\right]^{(n)} = \frac{2^n\,\Pi(\mu+n-1)}{\Pi(\mu-1)} C_{r-n}^{\mu+n}(y)$$

ist, so erhält man die folgenden Formeln

$$\sum_{n=0}^{n=\infty} (-1)^n \frac{y^{-\frac{n}{2}} J^{\mu+n}(\sqrt{y})\, C_n^\nu(x)}{2^{2n}\,\Pi\left(n+\nu-\frac{1}{2}\right)} = \frac{y^{\frac{\mu}{2}}}{\Pi(\nu-1)\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \cos^{2\nu-1}\varphi \left\{\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}} .\right.$$

$$. J^\mu\left(\sqrt{y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right) + \left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}} J^\mu\left.\left(\sqrt{y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\} d\varphi$$

$$= \frac{y^{\frac{\mu}{2}}}{\Pi(\nu-1)\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \left\{\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}} J^\mu\left(\sqrt{y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}}\right) +\right.$$

$$+ \left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}} J^\mu\left.\left(\sqrt{y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\} d\varphi$$

$$J^{\mu+n}(\sqrt{y}) = (-1)^n \frac{2^{2n+1}(n+\nu)\,\Pi(n)\,\Pi\left(n+\nu-\frac{1}{2}\right) y^{\frac{\mu+n}{2}}}{\Pi(n+2\nu-1)\,\Pi(\nu-1)\sqrt{\pi}} \left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1}\int_0^{\frac{\pi}{2}} (1-x^2)^{\frac{2\nu-1}{2}}\, C_n^\nu(x) .$$

$$. \cos^{2\nu-1}\varphi \left\{\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}} J^\mu\left(\sqrt{y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right) + \left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}\right.$$

$$J^\mu\left.\left(\sqrt{y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\} dx\, d\varphi$$

$$= (-1)^n \frac{2^{2n+1}(n+\nu)\,\Pi(n)\,\Pi\left(n+\nu-\frac{1}{2}\right) y^{\frac{\mu+n}{2}}}{\Pi(n+2\nu-1)\,\Pi(\nu-1)\sqrt{\pi}} \left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1}\int_0^{\frac{\pi}{2}} (1-x^2)^{\frac{2\nu-1}{2}}\, C_n^\nu(x) .$$

$$. \sin^{2\nu-1}\varphi \left\{\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}} J^\mu\left(\sqrt{y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}}\right) + \left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}\right.$$

$$J^\mu\left.\left(\sqrt{y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\} dx\, d\varphi$$

$$\sum_{n=0}^{n=\infty} (-1)^n \frac{y^{-\frac{n}{2}} J^{\mu+n}(\sqrt{y})\, C_{2n}^\nu(x)}{2^n\,\Pi\left(\frac{2n-1}{2}\right)\Pi\left(2n+\nu-\frac{1}{2}\right)} = \frac{y^{\frac{\mu}{2}}}{\Pi(\nu-1)} \int_0^{\frac{\pi}{2}} \cos^{2\nu-1}\varphi \left\{\left(y+x^2\cos^4\varphi+xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}} .\right.$$

$$. J^\mu\left(\sqrt{y+x^2\cos^4\varphi+xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right) + \left(y+x^2\cos^4\varphi-xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}} .$$

$$. J^\mu\left.\left(\sqrt{y+x^2\cos^4\varphi-xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)\right\} d\varphi$$

$$= \frac{y^{\frac{\mu}{2}}}{\Pi(\nu-1)} \int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \left\{\left(y+x^2\sin^4\varphi+xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}} .\right.$$

$$. J^\mu\left(\sqrt{y+x^2\sin^4\varphi+xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right) + \left(y+x^2\sin^4\varphi-xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}} .$$

$$. J^\mu\left.\left(\sqrt{y+x^2\sin^4\varphi-xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)\right\} d\varphi$$

$$\sum_{n=0}^{n=\infty}(-1)^n\frac{y^{-\frac{n}{2}}C_{2n+1}^\nu(x)J^{\mu+n}(\sqrt{y})}{2^n\Pi\left(\frac{2n+1}{2}\right)\Pi\left(2n+\nu+\frac{1}{2}\right)}=\frac{y^{\frac{\mu}{2}}}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{(x\cos\varphi+i\sin\varphi)\right.$$

$$.\left(y+x^2\cos^4\varphi+xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}}J^\mu\left(\sqrt{y+x^2\cos^4\varphi+xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)+$$

$$+(x\cos\varphi-i\sin\varphi)\left(y+x^2\cos^4\varphi-xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}}J^\mu\left(\sqrt{y+x^2\cos^4\varphi-xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)\Big\}\,d\varphi$$

$$=\frac{y^{\frac{\mu}{2}}}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{(x\sin\varphi+i\cos\varphi)\left(y+x^2\sin^4\varphi+xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}}\right..$$

$$.J^\mu\left(\sqrt{y+x^2\sin^4\varphi+xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)+(x\sin\varphi-i\cos\varphi)\left(y+x^2\sin^4\varphi-xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}}$$

$$J^\mu\left(\sqrt{y+x^2\sin^4\varphi-xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)\Big\}\,d\varphi$$

$$J^{\mu+n}(\sqrt{y})=(-1)^n\frac{2^{3n-1}(2n+\nu)\Pi(2n)\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)}{\Pi(2n+2\nu-1)\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 y^{\frac{\mu+n}{2}}.$$

$$.\int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}}C_{2n}^\nu(x)\,dx\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\,d\varphi\left\{\left(y+x^2\cos^4\varphi+xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}}\right..$$

$$.J^\mu\left(\sqrt{y+x^2\cos^4\varphi+xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)+\left(y+x^2\cos^4\varphi-xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}}.$$

$$.J^\mu\left(\sqrt{y+x^2\cos^4\varphi-xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)\Big\}$$

$$=(-1)^n\frac{2^{3n-1}(2n+\nu)\Pi(2n)\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)}{\Pi(\nu-1)\Pi(2n+2\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 y^{\frac{\mu+n}{2}}\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}}C_{2n}^\nu(x).$$

$$.\sin^{2\nu-1}\varphi\left\{\left(y+x^2\sin^4\varphi+xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}}J^\mu\left(\sqrt{y+x^2\sin^4\varphi+xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)+\right.$$

$$+\left(y+x^2\sin^4\varphi-xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}}J^\mu\left(\sqrt{y+x^2\sin^4\varphi-xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)\Big\}\,dx\,d\varphi$$

$$J^{\mu+n}(\sqrt{y})=(-1)^n\frac{2^{3n}(2n+\nu+1)\Pi(2n+1)\Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)}{\Pi(\nu-1)\Pi(2n+2\nu)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 y^{\frac{\mu+n}{2}}.$$

$$.\int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}}C_{2n+1}^\nu(x)\,dx\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\,d\varphi\left\{(x\cos\varphi+i\sin\varphi).\left(y+x^2\cos^4\varphi+xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}}\right..$$

$$.J^\mu\left(\sqrt{y+x^2\cos^4\varphi+xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)+(x\cos\varphi-i\sin\varphi)\left(y+x^2\cos^4\varphi-xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}}.$$

$$.J^\mu\left(\sqrt{y+x^2\cos^4\varphi-xi\cos\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)\Big\}$$

$$= (-1)^n \frac{2^{3n}(2n+\nu+1)\,\Pi(2n+1)\,\Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)}{\Pi(\nu-1)\,\Pi(2n+2\nu)} \left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 y^{\frac{\mu+n}{2}}.$$

$$.\int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}} C_{2n+1}^{\nu}(x)\,dx \int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi\,d\varphi\,\Big\{(x\sin\varphi+i\cos\varphi)\left(y+x^2\sin^4\varphi+xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}}.$$

$$J^\mu\left(\sqrt{y+x^2\sin^4\varphi+xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)+(x\sin\varphi-i\cos\varphi)\left(y+x^2\sin^4\varphi-xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{-\frac{\mu}{2}}.$$

$$.J^\mu\left(\sqrt{y+x^2\sin^4\varphi-xi\sin\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)\Big\}$$

$$\sum_{n=0}^{n=\infty}(-1)^n \frac{\Pi(n+2\nu-1)\,y^{-\frac{n}{2}} J^{\mu+n}(\sqrt{y})\,z^n}{2^{2n}\Pi(n)\,\Pi\left(n+\nu-\frac{1}{2}\right)(n+\nu)} = \frac{2y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1}\int_0^{\frac{\pi}{2}}\frac{(1-x^2)^{\frac{2\nu-1}{2}}}{(1-2zx+z^2)^\nu}.$$

$$.\cos^{2\nu-1}\varphi\,\Big\{\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}} J^\mu\left(\sqrt{y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}.$$

$$.J^\mu\left(\sqrt{y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\Big\}\,dx\,d\varphi$$

$$= \frac{2y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}dx}{(1-2zx+z^2)^\nu}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\,d\varphi\,\Big\{\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}.$$

$$.J^\mu\left(\sqrt{y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}} J^\mu\left(\sqrt{y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\Big\}$$

$$\sum_{n=0}^{n=\infty}(-1)^n \frac{\Pi(n+2\nu-1)\,y^{-\frac{\mu}{2}} J^{\mu+n}(\sqrt{y})\,z^n}{2^{2n}\Pi(n)\,\Pi\left(n+\nu-\frac{1}{2}\right)} = \frac{2\nu y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 (1-z^2)\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}\frac{(1-x^2)^{\frac{2\nu-1}{2}}}{(1-2zx+z^2)^{\nu+1}}.$$

$$.\cos^{2\nu-1}\varphi\,\Big\{\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}} J^\mu\left(\sqrt{y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}.$$

$$.J^\mu\left(\sqrt{y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\Big\}\,dx\,d\varphi$$

$$= \frac{2\nu y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 (1-z^2)\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}dx}{(1-2zx+z^2)^\nu}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\,d\varphi\,\Big\{\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}.$$

$$.J^\mu\left(\sqrt{y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}} J^\mu\left(\sqrt{y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\Big\}$$

$$\sum_{n=0}^{n=\infty}(-1)^n.$$

$$.\frac{\Pi(\alpha+n-1)\Pi(\beta+n-1)\Pi(n+2\nu-1)y^{-\frac{n}{2}}J^{\mu+n}(\sqrt{y})c^n F\left(\frac{\alpha+n}{2},\frac{\alpha+n+1}{2},\frac{\beta+n}{2},\frac{\beta+n+1}{2},n+\nu+1,\frac{\gamma+n}{2},\frac{\gamma+n+1}{2},c^2\right)}{2^{3n+1}\Pi(n)\,\Pi(n+\nu)\,\Pi(\gamma+n-1)\,\Pi\left(n+\nu-\frac{1}{2}\right)} =$$

$$= \frac{\Pi(\alpha-1)\Pi(\beta-1)y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\gamma-1)[\Pi(\nu-1)]^2}\int_{-1}^{+1} F(\alpha,\beta,\gamma,cx)(1-x^2)^{\frac{2\nu-1}{2}}dx\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}.\right.$$

$$\left. J^{\mu}\left(\sqrt{y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}J^{\mu}\left(\sqrt{y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\}d\varphi$$

$$= \frac{\Pi(\alpha-1)\Pi(\beta-1)y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\gamma-1)[\Pi(\nu-1)]^2}\int_{-1}^{+1} F(\alpha,\beta,\gamma,cx)(1-x^2)^{\frac{2\nu-1}{2}}dx\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}.\right.$$

$$\left. .J^{\mu}\left(\sqrt{y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}J^{\mu}\left(\sqrt{y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=\infty}\frac{y^{-\frac{n}{2}}J^{\mu+n}(\sqrt{y})\,C_n^\nu(\cos x)}{2^n\,\Pi\left(\frac{2n-1}{2}\right)\Pi\left(2n+2\nu-\frac{1}{2}\right)} = \frac{\Pi\left(\frac{2\nu-1}{2}\right)y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\sin^{2\nu-1}\frac{x}{2}}\int_0^x\int_0^{\frac{\pi}{2}}(\cos\varphi-\cos x)^{\nu-1}\cos\frac{\varphi}{2}\,.$$

$$\cdot\cos^{4\nu-1}\psi\left\{\left(y+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}}.\right.$$

$$.J^{\mu}\left(\sqrt{y+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)+\left(y+\sin\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}}.$$

$$\left.\cdot J^{\mu}\left(\sqrt{y+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\varphi-\frac{\sin^2 2\psi}{4}}\right)\right\}d\varphi\,d\psi$$

$$= \frac{\Pi\left(\frac{2\nu-1}{2}\right)y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\sin^{2\nu-1}\frac{x}{2}}\int_0^x(\cos\varphi-\cos x)^{\nu-1}\cos\frac{\varphi}{2}\,d\varphi\int_0^{\frac{\pi}{2}}\sin^{4\nu-1}\psi\left\{\left(y+\sin^2\frac{\varphi}{2}\sin^4\psi+\right.\right.$$

$$\left.+i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}}J^{\mu}\left(\sqrt{y+\sin^2\frac{\varphi}{2}\sin^4\psi+i\sin\frac{\varphi}{2}\sin^2\varphi\,2\psi-\frac{\sin^2 2\psi}{4}}\right)+$$

$$\left.+\left(y+\sin^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}}.J^{\mu}\left(\sqrt{y+\sin^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin^2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\}d\psi$$

$$\sum_{n=0}^{n=\infty}(-1)^n\frac{y^{-\frac{n}{2}}J^{\mu+n}(\sqrt{y})\,C_n^\nu(\cos x)}{2^n\,\Pi\left(\frac{2n-1}{2}\right)\Pi\left(2n+2\nu-\frac{1}{2}\right)} = \frac{\Pi\left(\frac{2\nu-1}{2}\right)y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\nu-1)\Pi(2\nu-1)\cos^{2\nu-1}\frac{x}{2}}\int_x^\pi\int_0^{\frac{\pi}{2}}(\cos x-\cos\varphi)^{\nu-1}\sin\frac{\varphi}{2}\,.$$

$$\cdot\cos^{4\nu-1}\psi\left\{\left(y+\cos^2\frac{\varphi}{2}\cos^4\psi+i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}}.\right.$$

$$.J^{\mu}\left(\sqrt{y+\cos^2\frac{\varphi}{2}\cos^4\psi+i\cos\frac{\varphi}{2}\cos^2\psi\sin^2\psi-\frac{\sin^2 2\psi}{4}}\right)+\left(y+\cos^2\frac{\varphi}{2}\cos^4\psi-i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}}.$$

$$\left.\cdot J^{\mu}\left(\sqrt{y+\cos^2\frac{\varphi}{2}\cos^4\psi-i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\}d\varphi\,d\psi$$

$$= \frac{\Pi\left(\frac{2\nu-1}{2}\right) y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\cos^{2\nu-1}\frac{x}{2}} \int_x^\pi (\cos x-\cos\varphi)^{\nu-1}\sin\frac{\varphi}{2}\,d\varphi \int_0^{\frac{\pi}{2}} \sin^{4\nu-1}\psi \left\{\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi+\right.\right.$$

$$\left. + i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}} J^\mu\left(\sqrt{y+\cos^2\frac{\varphi}{2}\sin^4\psi+i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)+$$

$$\left.+\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}} J^\mu\left(\sqrt{y+\cos^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\} d\psi$$

$$\sum_{n=0}^{n=\infty} \frac{y^{-\frac{n}{2}} J^{\mu+n}(\sqrt{y})\, C^\nu_{2n+1}(\cos x)}{2^n\,\Pi\left(\frac{2n+1}{2}\right)\Pi\left(2n+2\nu+\frac{1}{2}\right)} = \frac{\Pi\left(\frac{2\nu-3}{2}\right) y^{\frac{\mu}{2}}}{\Pi(\nu-2)\,\Pi(2\nu-2)\sqrt{\pi}\sin^{2\nu-1}\frac{x}{2}} \int_0^x\int_0^{\frac{\pi}{2}} (\cos\varphi-\cos x)^{\nu-2}\sin\varphi\,.$$

$$.\cos^{4\nu-3}\psi\left\{\left(\sin\frac{\varphi}{2}\cos\psi+i\sin\psi\right)\left(y+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}}.\right.$$

$$.J^\mu\left(\sqrt{y+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)+\left(\sin\frac{\varphi}{2}\cos\psi-i\sin\psi\right).$$

$$\left..\left(y+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}} J^\mu\left(\sqrt{y+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\} d\varphi\, d\psi$$

$$= \frac{\Pi\left(\frac{2\nu-3}{2}\right) y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\nu-2)\,\Pi(2\nu-2)\sin^{2\nu-1}\frac{x}{2}} \int_0^x (\cos\varphi-\cos x)^{\nu-2}\sin\varphi\, d\varphi \int_0^{\frac{\pi}{2}} \sin^{4\nu-3}\psi\left\{\left(\sin\frac{\varphi}{2}\sin\psi+i\cos\psi\right).\right.$$

$$.\left(y+\sin^2\frac{\varphi}{2}\sin^4\psi+i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}} J^\mu\left(\sqrt{y+\sin^2\frac{\varphi}{2}\sin^4\psi+i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)+$$

$$+\left(\sin\frac{\varphi}{2}\sin\psi-i\cos\psi\right)\left(y+\sin^2\frac{\varphi}{2}\sin^4\psi+i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}}.$$

$$\left..J^\mu\left(\sqrt{y+\sin^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\} d\psi$$

$$\sum_{n=0}^{n=\infty}(-1)^n \frac{y^{-\frac{n}{2}} J^{\mu+n}(\sqrt{y})\, C^\nu_{2n+1}(\cos x)}{2^n\,\Pi\left(\frac{2n+1}{2}\right)\Pi\left(2n+2\nu+\frac{1}{2}\right)} = \frac{\Pi\left(\frac{2\nu-3}{2}\right) y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\nu-2)\,\Pi(2\nu-2)\cos^{2\nu-1}\frac{x}{2}} \int_x^\pi\int_0^{\frac{\pi}{2}} (\cos x-\cos\varphi)^{\nu-2}\sin\varphi\,.$$

$$.\cos^{4\nu-3}\psi\left\{\left(\cos\frac{\varphi}{2}\cos\psi+i\sin\psi\right)\left(y+\cos^2\frac{\varphi}{2}\cos^4\psi+i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}}\right.$$

$$.J^\mu\left(\sqrt{y+\cos^2\frac{\varphi}{2}\cos^4\psi+i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)+\left(\cos\frac{\varphi}{2}\cos\psi-i\sin\psi\right)\left(y+\cos^2\frac{\varphi}{2}\cos^4\psi-i\cos\frac{\varphi}{2}\cos^2\psi\,.\right.$$

$$\left..\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}} J^\mu\left(\sqrt{y+\cos^2\frac{\varphi}{2}\cos^4\psi-i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\} d\varphi\, d\psi$$

$$= -\frac{\Pi\left(\frac{2\nu-3}{2}\right) y^{\frac{\mu}{2}}}{\sqrt{\pi}\,\Pi(\nu-2)\,\Pi(2\nu-2)\cos^{2\nu-1}\frac{\pi}{2}} \int_x^{\pi} (\cos x-\cos\varphi)^{\nu-2}\sin\varphi\, d\varphi \int_0^{\frac{\pi}{2}} \sin^{4\nu-3}\psi \left\{\left(\cos\frac{\varphi}{2}\sin\psi+i\cos\psi\right)\right.$$

$$.\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi+i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}} J^{\mu}\left(\sqrt{y+\cos^2\frac{\varphi}{2}\sin^4\psi+i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)+$$

$$+\left(\cos\frac{\varphi}{2}\sin\psi-i\cos\psi\right)\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi-i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{-\frac{\mu}{2}}.$$

$$\left..J^{\mu}\left(\sqrt{y+\cos^2\frac{\varphi}{2}\sin^4\varphi-i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\} d\psi$$

$$\sum_{n=0}^{n=\infty}\frac{y^{-\frac{n}{2}}J^{\mu-n}(\sqrt{y})\,C_n^{\nu}(x)}{2^{2n}\,\Pi\left(n+\nu-\frac{1}{2}\right)} = \frac{1}{\sqrt{\pi}\,\Pi(\nu-1)\,y^{\frac{\mu}{2}}}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}}\right..$$

$$\left..J^{\mu}\left(\sqrt{y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\} d\varphi$$

$$= \frac{1}{\sqrt{\pi}\,\Pi(\nu-1)\,y^{\frac{\mu}{2}}}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\right.$$

$$\left.+\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\} d\varphi$$

$$J^{\mu-n}(\sqrt{y}) = \frac{2^{n-1}(n+\nu)\,\Pi(n)\,\Pi\left(n+\nu-\frac{1}{2}\right)y^{\frac{n-\mu}{2}}}{\Pi(n+2\nu-1)\,\Pi(\nu-1)\sqrt{\pi}}\left[\frac{\Pi(2\nu-1)}{2^{\nu}\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}}\,C_n^{\nu}(x)\,.$$

$$.\cos^{2\nu-1}\varphi\left\{\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}}\right..$$

$$\left..J^{\mu}\left(\sqrt{y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\} dx\, d\varphi$$

$$= \frac{2^{n-1}(n+\nu)\,\Pi(n)\,\Pi\left(n+\nu-\frac{1}{2}\right)y^{\frac{n-\mu}{2}}}{\Pi(n+2\nu-1)\,\Pi(\nu-1)\sqrt{\pi}}\left[\frac{\Pi(2\nu-1)}{2^{\nu}\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}}\,C_n^{\nu}(x)\,.$$

$$.\sin^{2\nu-1}\varphi\left\{\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}}\right..$$

$$\left.J^{\mu}\left(\sqrt{y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\} dx\, d\varphi$$

$$\sum_{n=0}^{n=\infty}\frac{y^{-\frac{n}{2}}J^{\mu-n}(\sqrt{y})\,C_{2n}^{\nu}(x)}{2^{n}\,\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)} = \frac{1}{\Pi(\nu-1)\,y^{\frac{\mu}{2}}}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{\left(y+x^2\cos^4\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}}\right..$$

$$.J^{\mu}\left(\sqrt{y+x^2\cos^4\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)+\left(y+x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}}.$$

$$\left.J^{\mu}\left(\sqrt{y+x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)\right\} d\varphi$$

$$= \frac{1}{\Pi(\nu-1)\, y^{\frac{\mu}{2}}} \int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \left\{ \left(y + x^2\sin^4\varphi + xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}} . \right.$$

$$\cdot J^{\mu}\left(\sqrt{y + x^2\sin^4\varphi + xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}}\right) + \left(y + x^2\sin^4\varphi - xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}} .$$

$$\left. \cdot J^{\mu}\left(\sqrt{y + x^2\sin^4\varphi - xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}}\right)\right\} d\varphi$$

$$\sum_{n=0}^{n=\infty} \frac{y^{-\frac{n}{2}} J^{\mu-n}(\sqrt{y})\, C^{\nu}_{2n+1}(x)}{2^n\, \Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)} = \frac{1}{\Pi(\nu-1)\, y^{\frac{\mu}{2}}} \int_0^{\frac{\pi}{2}} \cos^{2\nu-1}\varphi \left\{ (x\cos\varphi + i\sin\varphi) . \right.$$

$$\cdot \left(y + x^2\cos^4\varphi + xi\cos^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y + x^2\cos^4\varphi + xi\cos^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}}\right) + (x\cos\varphi - i\sin\varphi) .$$

$$\left. \cdot \left(y + x^2\cos^4\varphi - xi\cos^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y + x^2\cos^4\varphi - xi\cos^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}}\right)\right\} d\varphi$$

$$= \frac{1}{\Pi(\nu-1)\, y^{\frac{\mu}{2}}} \int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \left\{ (x\sin\varphi + i\cos\varphi) \left(y + x^2\sin^4\varphi + xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}} . \right.$$

$$\cdot J^{\mu}\left(\sqrt{y + x^2\sin^4\varphi + xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}}\right) + (x\sin\varphi - i\cos\varphi)\left(y + x^2\sin^4\varphi - xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}}$$

$$\left. . J^{\mu}\left(\sqrt{y + x^2\sin^4\varphi - xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}}\right)\right\} d\varphi$$

$$J^{\mu-n}(\sqrt{y}) = \frac{2^{3n-1}(2n+\nu)\,\Pi(2n)\,\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)}{\Pi(\nu-1)\,\Pi(2n+2\nu-1)\, y^{\frac{\mu-n}{2}}} \left[\frac{\Pi(2\nu-1)}{2^{\nu}\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1}\int_0^{\frac{\pi}{2}} (1-x^2)^{\frac{2\nu-1}{2}} C^{\nu}_{2n}(x) .$$

$$. \cos^{2\nu-1}\varphi \left\{ \left(y + x^2\cos^4\varphi + xi\cos^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y + x^2\cos^4\varphi + xi\cos^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}}\right) + \right.$$

$$\left. + \left(y + x^2\cos^4\varphi - xi\cos^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y + x^2\cos^4\varphi - xi\cos^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}}\right)\right\} dx\, d\varphi$$

$$= \frac{2^{3n-1}(2n+\nu)\,\Pi(2n)\,\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)}{\Pi(\nu-1)\,\Pi(2n+2\nu-1)\, y^{\frac{\mu-n}{2}}} \left[\frac{\Pi(2\nu-1)}{2^{\nu}\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1}\int_0^{\frac{\pi}{2}} (1-x^2)^{\frac{2\nu-1}{2}} C^{\nu}_{2n}(x) .$$

$$. \sin^{2\nu-1}\varphi \left\{ \left(y + x^2\sin^4\varphi + xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y + x^2\sin^4\varphi + xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}}\right) + \right.$$

$$\left. + \left(y + x^2\sin^4\varphi - xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y + x^2\sin^4\varphi - xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}}\right)\right\} dx\, d\varphi$$

$$J^{\mu-n}(\sqrt{y}) = \frac{2^{3n}(2n+\nu+1)\,\Pi(2n+1)\,\Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)}{\Pi(2n+2\nu)\,\Pi(\nu-1)\,y^{\frac{\mu-n}{2}}}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}}\,C_{2n+1}^\nu(x)\,.$$

$$.\cos^{2\nu-1}\varphi\left\{(x\cos\varphi+i\sin\varphi)\left(y+x^2\cos^4\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}}.\right.$$

$$.J^\mu\left(\sqrt{y+x^2\cos^4\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)+(x\cos\varphi-i\sin\varphi)\left(y+x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{2}\right)^{\frac{\mu}{2}}.$$

$$\left..J^\mu\left(\sqrt{y+x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)\right\}dx\,d\varphi$$

$$= \frac{2^{3n}(2n+\nu+1)\,\Pi(2n+1)\,\Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)}{\Pi(2n+2\nu)\,\Pi(\nu-1)\,y^{\frac{\mu-n}{2}}}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}}\,C_{2n+1}^\nu(x)\,.$$

$$.\sin^{2\nu-1}\varphi\left((x\sin\varphi+i\cos\varphi)\left\{\left(y+x^2\sin^4\varphi+xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}}.\right.\right.$$

$$.J^\mu\left(\sqrt{y+x^2\sin^4\varphi+xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)+(x\sin\varphi-i\cos\varphi)\left(y+x^2\sin^4\varphi-xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)^{\frac{\mu}{2}}.$$

$$\left..J^\mu\left(\sqrt{y+x^2\sin^4\varphi-xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}}\right)\right\}dx\,d\varphi$$

$$\sum_{n=0}^{n=\infty}\frac{\Pi(n+2\nu-1)\,y^{-\frac{n}{2}}\,J^{\mu-n}(\sqrt{y})\,z^n}{2^{2n}\,\Pi(n)\,\Pi\left(n+\nu-\frac{1}{2}\right)(n+\nu)} = \frac{2}{\sqrt{\pi}\,\Pi(\nu-1)\,y^{\frac{\mu}{2}}}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}\frac{(1-x^2)^{\frac{2\nu-1}{2}}}{(1-2zx+z^2)^\nu}.$$

$$.\cos^{2\nu-1}\varphi\left\{\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}}J^\mu\left(\sqrt{y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}}.\right.$$

$$\left..J^\mu\left(\sqrt{y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\}dx\,d\varphi$$

$$= \frac{2}{\sqrt{\pi}\,\Pi(\nu-1)\,y^{\frac{\mu}{2}}}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^\nu}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\,d\varphi\left\{\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}}.\right.$$

$$\left..J^\mu\left(\sqrt{y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}}J^\mu\left(\sqrt{y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\}$$

$$\sum_{n=0}^{n=\infty}\frac{\Pi(n+2\nu-1)\,y^{-\frac{n}{2}}\,J^{\mu-n}(\sqrt{y})\,z^n}{2^{2n}\,\Pi(n)\,\Pi\left(n+\nu-\frac{1}{2}\right)} = \frac{2\nu}{\sqrt{\pi}\,\Pi(\nu-1)\,y^{\frac{\mu}{2}}}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}\frac{(1-x^2)^{\frac{2\nu-1}{2}}}{(1-2zx+z^2)^{\nu+1}}.$$

$$\cos^{2\nu-1}\varphi\left\{\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}}J^\mu\left(\sqrt{y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}}\right.$$

$$\left..J^\mu\left(\sqrt{y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\}dx\,d\varphi$$

$$= \frac{2\nu}{\sqrt{\pi}\,\Pi(\nu-1)\,y^{\frac{\mu}{2}}} \left[\frac{\Pi(2\nu-1)}{2^{\nu}\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1} \frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^{\nu+1}} \int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi\,d\varphi \left\{\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}} .\right.$$

$$\left. .J^{\mu}\left(\sqrt{y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}}\right) + \left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\}$$

$$\sum_{n=0}^{n=\infty} \frac{\Pi(\alpha+n-1)\,\Pi(\beta+n-1)\,\Pi(n+2\nu-1)\,y^{-\frac{n}{2}}\,J^{\mu-n}(\sqrt{y})\,c^n}{2^{3n+1}\,\Pi(n)\,\Pi(n+\nu)\,\Pi(\gamma+n-1)\,\Pi\left(n+\nu-\frac{1}{2}\right)} .$$

$$.F\left(\frac{\alpha+n}{2}, \frac{\alpha+n+1}{2}, \frac{\beta+n}{2}, \frac{\beta+n+1}{2}, n+\nu+1, \frac{\gamma+n}{2}, \frac{\gamma+n+1}{2}, c^2\right) = \frac{\Pi(\alpha-1)\,\Pi(\beta-1)}{\sqrt{\pi}\,\Pi(\gamma-1)\,[\Pi(\nu-1)]^2\,y^{\frac{\mu}{2}}} .$$

$$.\int_{-1}^{+1} F(\alpha,\beta,\gamma,cx)\,(1-x^2)^{\frac{2\nu-1}{2}}dx \int_0^{\frac{\pi}{2}} \cos^{2\nu-1}\varphi \left\{\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right) + \right.$$

$$\left. + \left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\} d\varphi$$

$$= \frac{\Pi(\alpha-1)\,\Pi(\beta-1)}{\sqrt{\pi}\,\Pi(\gamma-1)\,[\Pi(\nu-1)]^2\,y^{\frac{\mu}{2}}} \int_{-1}^{+1} F(\alpha,\beta,\gamma,cx)\,(1-x^2)^{\frac{2\nu-1}{2}}dx \int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \left\{\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}} .\right.$$

$$\left. .J^{\mu}\left(\sqrt{y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}}\right) + \left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{\frac{\mu}{2}} J^{\mu}\left(\sqrt{y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\} d\varphi$$

$$\sum_{n=0}^{n=\infty} (-1)^n \frac{y^{-\frac{n}{2}}\,J^{\mu-n}(\sqrt{y})\,C_n^{\nu}(\cos x)}{2^n\,\Pi\left(\frac{2n-1}{2}\right)\Pi\left(2n+2\nu-\frac{1}{2}\right)} = \frac{\Pi\left(\frac{2\nu-1}{2}\right)}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\,y^{\frac{\mu}{2}}\sin^{2\nu-1}\frac{x}{2}} .$$

$$.\int_0^{x} (\cos\varphi-\cos x)^{\nu-1}\cos\frac{\varphi}{2}\,dx \int_0^{\frac{\pi}{2}} \cos^{4\nu-1}\psi\,d\psi \left\{\left(y+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}} .\right.$$

$$.J^{\mu}\left(\sqrt{y+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right) + \left(y+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}} .$$

$$\left. .J^{\mu}\left(\sqrt{y+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\}$$

$$= \frac{\Pi\left(\frac{2\nu-1}{2}\right)}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\,y^{\frac{\mu}{2}}\sin^{2\nu-1}\frac{x}{2}} \int_0^{x}\int_0^{\frac{\pi}{2}} (\cos\varphi-\cos x)^{\nu-1}\cos\frac{\varphi}{2} .$$

$$.\sin^{4\nu-1}\psi \left\{\left(y+\sin^2\frac{\varphi}{2}\sin^4\psi+i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}}\right.$$

$$.J^{\mu}\left(\sqrt{y+\sin^2\frac{\varphi}{2}\sin^4\psi+i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right) + \left(y+\sin^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}} .$$

$$\left. .J^{\mu}\left(\sqrt{y+\sin^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\} d\varphi\,d\psi$$

$$\sum_{n=0}^{n=\infty} \frac{y^{-\frac{n}{2}} J^{\mu-n}(\sqrt{y})\, C_n^\nu(\cos x)}{2^n\, \Pi\left(\frac{2n-1}{2}\right) \Pi\left(2n+2\nu-\frac{1}{2}\right)} = \frac{\Pi\left(\frac{2\nu-1}{2}\right)}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\, y^{\frac{\mu}{2}} \cos^{2\nu-1}\frac{x}{2}} \int_x^{\pi}\int_0^{\frac{\pi}{2}} (\cos x-\cos\varphi)^{\nu-1} \sin\frac{\varphi}{2}\,.$$

$$.\cos^{4\nu-1}\psi \left\{\left(y+\cos^2\frac{\varphi}{2}\cos^4\psi+i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}}.\right.$$

$$.J^\mu\left(\sqrt{y+\cos^2\frac{\varphi}{2}\cos^4\psi+i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)+\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi-i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}}.$$

$$\left..J^\mu\left(\sqrt{y+\cos^2\frac{\varphi}{2}\cos^4\psi-i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\} d\varphi\, d\psi$$

$$= \frac{\Pi\left(\frac{2\nu-1}{2}\right)}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\, y^{\frac{\mu}{2}} \cos^{2\nu-1}\frac{x}{2}} \int_x^{\pi}\int_0^{\frac{\pi}{2}} (\cos x-\cos\varphi)^{\nu-1} \sin\frac{\varphi}{2}\,.$$

$$.\sin^{4\nu-1}\psi \left\{\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi+i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}}.\right.$$

$$.J^\mu\left(\sqrt{y+\cos^2\frac{\varphi}{2}\sin^4\psi+i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)+\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi-i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}}.$$

$$\left..J^\mu\left(\sqrt{y+\cos^2\frac{\varphi}{2}\sin^4\psi-i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\} d\varphi\, d\psi$$

$$\sum_{n=0}^{n=\infty} (-1)^n \frac{y^{-\frac{n}{2}} J^{\mu-n}(\sqrt{y})\, C_n^\nu(\cos x)}{2^n\, \Pi\left(\frac{2n+1}{2}\right) \Pi\left(2n+2\nu+\frac{1}{2}\right)} = \frac{\Pi\left(\frac{2\nu-3}{2}\right)}{\sqrt{\pi}\,\Pi(\nu-2)\,\Pi(2\nu-2)\, y^{\frac{\mu}{2}} \sin^{2\nu-1}\frac{x}{2}}\,.$$

$$.\int_0^x (\cos\varphi-\cos x)^{\nu-2}\sin\varphi\, d\varphi \int_0^{\frac{\pi}{2}} \cos^{4\nu-3}\psi \left\{\left(\sin\frac{\varphi}{2}\cos\psi+i\sin\psi\right).\right.$$

$$\left(y+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}} J^\mu\left(\sqrt{y+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)+$$

$$+\left(\sin\frac{\varphi}{2}\cos\psi-i\sin\psi\right)\left(y+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}}.$$

$$\left..J^\mu\left(\sqrt{y+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\} d\psi$$

$$= \frac{\Pi\left(\frac{2\nu-3}{2}\right)}{\sqrt{\pi}\,\Pi(\nu-2)\,\Pi(2\nu-2)\, y^{\frac{\mu}{2}} \sin^{2\nu-1}\frac{x}{2}} \int_0^x (\cos\varphi-\cos x)^{\nu-2}\sin\varphi\, d\varphi \int_0^{\frac{\pi}{2}} \sin^{4\nu-3}\psi \left\{\left(\sin\frac{\varphi}{2}\sin\psi+i\cos\psi\right).\right.$$

$$.\left(y+\sin^2\frac{\varphi}{2}\sin^4\psi+i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}} J^\mu\left(\sqrt{y+\sin^2\frac{\varphi}{2}\sin^4\psi+i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)+$$

$$+\left(\sin\frac{\varphi}{2}\sin\psi-i\cos\psi\right)\left(y+\sin^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}}.$$

$$\left..J^\mu\left(\sqrt{y+\sin^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\right\} d\psi$$

$$\sum_{n=0}^{n=\infty} \frac{y^{-\frac{n}{2}} J^{\mu-n}(\sqrt{y})\, C_n^\nu(\cos x)}{2^n \Pi\left(\frac{2n+1}{2}\right) \Pi\left(2n+2\nu+\frac{1}{2}\right)} = \frac{\Pi\left(\frac{2\nu-3}{2}\right)}{\sqrt{\pi}\, \Pi(\nu-2)\, \Pi(2\nu-2)\, y^{\frac{\mu}{2}} \cos^{2\nu-1}\frac{x}{2}} \int_x^\pi \int_0^{\frac{\pi}{2}} (\cos x-\cos\varphi)^{\nu-2} \sin\varphi\ .$$

$$.\ \cos^{4\nu-3}\psi \left\{\left(\cos\frac{\varphi}{2}\cos\psi+i\sin\psi\right)\left(y+\cos^2\frac{\varphi}{2}\cos^4\psi+i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}}.\right.$$

$$.\, J^\mu\left(\sqrt{y+\cos^2\frac{\varphi}{2}\cos^4\psi+i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)+\left(\cos\frac{\varphi}{2}\cos\psi-i\sin\psi\right).$$

$$.\left(y+\cos^2\frac{\varphi}{2}\cos^4\psi-i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}} J^\mu\left(\sqrt{y+\cos^2\frac{\varphi}{2}\cos^4\psi-i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\Bigg\} d\varphi d\psi$$

$$= \frac{\Pi\left(\frac{2\nu-3}{2}\right)}{\sqrt{\pi}\, \Pi(\nu-2)\, \Pi(2\nu-2)\, y^{\frac{\mu}{2}} \cos^{2\nu-1}\frac{x}{2}} \int_x^\pi (\cos x-\cos\varphi)^{\nu-2} \sin\varphi\, d\varphi \int_0^{\frac{\pi}{2}} \sin^{4\nu-3}\psi \left\{\left(\cos\frac{\varphi}{2}\sin\psi+i\cos\psi\right).\right.$$

$$.\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi+i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}} J^\mu\left(\sqrt{y+\cos^2\frac{\varphi}{2}\sin^4\psi+i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)+$$

$$+\left(\cos\frac{\varphi}{2}\sin\psi-i\cos\psi\right)\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi-i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)^{\frac{\mu}{2}}.$$

$$.\, J^\mu\left(\sqrt{y+\cos^2\frac{\varphi}{2}\sin^4\psi-i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}}\right)\Bigg\} d\psi$$

$$\sum_{n=0}^{n=r} \frac{\Pi(\mu+n-1)}{\Pi\left(n+\nu-\frac{1}{2}\right)} C_{r-n}^{\mu+n}(y)\, C_n^\nu(x) =$$

$$= \frac{\Pi(\mu-1)}{\sqrt{\pi}\, \Pi(\nu-1)} \int_0^{\frac{\pi}{2}} \cos^{2\nu-1}\varphi \left\{C_r^\mu\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C_r^\mu\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\} d\varphi$$

$$= \frac{\Pi(\mu-1)}{\sqrt{\pi}\, \Pi(\nu-1)} \int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \left\{C_r^\mu\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C_r^\mu\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\} d\varphi$$

$$C_{r-n}^{\mu+n}(y) = \frac{(n+\nu)\, \Pi(\mu-1)\, \Pi(n)\, \Pi\left(n+\nu-\frac{1}{2}\right)}{2\Pi(n+\mu-1)\, \Pi(n+2\nu-1)\, \Pi(\nu-1)\sqrt{\pi}} \left[\frac{\Pi(2\nu-1)}{2^\nu\, \Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1}\int_0^{\frac{\pi}{2}} (1-x^2)^{\frac{2\nu-1}{2}} C_n^\nu(x)\ .$$

$$.\ \cos^{2\nu-1}\varphi \left\{C_r^\mu\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C_r^\mu\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\} dx\, d\varphi$$

$$= \frac{(n+\nu)\, \Pi(\mu-1)\, \Pi(n)\, \Pi\left(n+\nu-\frac{1}{2}\right)}{2\Pi(n+\mu-1)\, \Pi(n+2\nu-1)\, \Pi(\nu-1)\sqrt{\pi}} \left[\frac{\Pi(2\nu-1)}{2^\nu\, \Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1}\int_0^{\frac{\pi}{2}} (1-x^2)^{\frac{2\nu-1}{2}} C_n^\nu(x)\ .$$

$$.\ \sin^{2\nu-1}\varphi \left\{C_r^\mu\left(y+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C_r^\mu\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\} dx\, d\varphi$$

$$\sum_{n=0}^{n=r} \frac{2^n \Pi(n+\mu-1)}{\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)} C_{r-n}^{\mu+n}(y)\, C_{2n}^\nu(x) = \frac{\Pi(\mu-1)}{\Pi(\nu-1)}\, .$$

$$.\int_0^{\frac{\pi}{2}} \cos^{2\nu-1}\varphi \left\{ C_r^\mu\left(y+x^2\cos^4\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+C_r^\mu\left(y+x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\} d\varphi$$

$$=\frac{\Pi(\mu-1)}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \left\{ C_r^\mu\left(y+x^2\sin^4\varphi+xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\right.$$

$$\left.+\,C_r^\mu\left(y+x^2\sin^4\varphi-xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\} d\varphi$$

$$\sum_{n=0}^{n=r} \frac{2^n \Pi(n+\mu-1)}{\Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)} C_{r-n}^{\mu+n}(y)\, C_{2n+1}^\nu(x) = \frac{\Pi(\mu-1)}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}} \cos^{2\nu-1}\varphi \left\{(x\cos\varphi+i\sin\varphi)\,.\right.$$

$$.\,C_r^\mu\left(y+x^2\cos^4\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+(x\cos\varphi-i\sin\varphi)\,C_r^\mu\left(y+x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\} d\varphi$$

$$=\frac{\Pi(\mu-1)}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \left\{(x\sin\varphi+i\cos\varphi)\, C_r^\mu\left(y+x^2\sin^4\varphi+xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\right.$$

$$\left.+\,(x\sin\varphi-i\cos\varphi)\, C_r^\mu\left(y+x^2\sin^4\varphi-xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\} d\varphi$$

$$C_{r-n}^{\mu+n}(y)=\frac{2^{n-1}\Pi(\mu-1)(2n+\nu)\Pi(2n)\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)}{\Pi(\nu-1)\Pi(2n+2\nu-1)\Pi(n+\mu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}} C_{2n}^\nu(x).$$

$$.\,\cos^{2\nu-1}\varphi\left\{C_r^\mu\left(y+x^2\cos^2\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+C_r^\mu\left(y+x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\} dx\, d\varphi$$

$$=\frac{2^{n-1}\Pi(\mu-1)(2n+\nu)\Pi(2n)\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)}{\Pi(\nu-1)\Pi(2n+2\nu-1)\Pi(n+\mu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\int_0^{\frac{\pi}{2}}(1-x^2)^{\frac{2\nu-1}{2}} C_{2n}^\nu(x).$$

$$.\,\sin^{2\nu-1}\varphi\left\{C_r^\mu\left(y+x^2\sin^4\varphi+xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+C_r^\mu\left(y+x^2\sin^4\varphi-xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\} dx\, d\varphi$$

$$C_{r-n}^{\mu+n}(y)=\frac{2^n\Pi(\mu-1)(2n+\nu+1)\Pi(2n+1)\Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)}{\Pi(\nu-1)\Pi(2n+2\nu)\Pi(n+\mu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 .$$

$$.\int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}} C_{2n+1}^\nu(x)\,dx\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{(x\cos\varphi+i\sin\varphi)\,C_r^\mu\left(y+x^2\cos^4\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\right.$$

$$\left.+\,(x\cos\varphi-i\sin\varphi)\,C_r^\mu\left(y+x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\} d\varphi$$

$$= \frac{2^n \Pi(\mu-1)(2n+\nu+1)\Pi(2n+1)\Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)}{\Pi(\nu-1)\Pi(2n+2\nu)\Pi(n+\mu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu \Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1}\int_0^{\frac{\pi}{2}} (1-x^2)^{\frac{2\nu-1}{2}} C_{2n+1}^{\nu}(x).$$

$$. \sin^{2\nu-1}\varphi \left\{(x\sin\varphi+i\cos\varphi)\, C_r^\mu\left(y+x^2\sin^4\varphi+xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\right.$$

$$\left.+(x\sin\varphi-i\cos\varphi)\, C_r^\mu\left(y+x^2\sin^4\varphi-xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\} dx\, d\varphi$$

$$\sum_{n=0}^{n=r} \frac{\Pi(\mu+n-1)\Pi(n+2\nu-1)}{\Pi(n)\Pi\left(n+\nu-\frac{1}{2}\right)(n+\nu)} C_{r-n}^{\mu+n}(y) z^n = \frac{2\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1}\int_0^{\frac{\pi}{2}} \frac{(1-x^2)^{\frac{2\nu-1}{2}}}{(1-2zx+z^2)^\nu}\,.$$

$$. \cos^{2\nu-1}\varphi\left\{C_r^\mu\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C_r^\mu\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\} dx\, d\varphi$$

$$= \frac{2\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 \int_{-1}^{+1} \frac{(1-x^2)^{\frac{2\nu-1}{2}}dx}{(1-2zx+z^2)^\nu}\int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi\left\{C_r^\mu\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)+\right.$$

$$\left.+C_r^\mu\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\} d\varphi$$

$$\sum_{n=0}^{n=r} \frac{\Pi(\mu+n-1)\Pi(n+2\nu-1)}{\Pi(n)\Pi\left(n+\nu-\frac{1}{2}\right)} C_{r-n}^{\mu+n}(y) z^n = \frac{2\nu\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 (1-z^2)\int_{-1}^{+1}\int_0^{\frac{\pi}{2}} \frac{(1-x^2)^{\frac{2\nu-1}{2}}}{(1-2zx+z^2)^{\nu+1}}\,.$$

$$. \cos^{2\nu-1}\varphi\left\{C_r^\mu\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C_r^\mu\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\} dx\, d\varphi$$

$$= \frac{2\nu\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 (1-z^2)\int_{-1}^{+1} \frac{(1-x^2)^{\frac{2\nu-1}{2}}dx}{(1-2zx+z^2)^{\nu+1}}\int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi\left\{C_r^\mu\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)+\right.$$

$$\left.+C_r^\mu\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\} d\varphi$$

$$\sum_{n=0}^{n=r} \frac{\Pi(\mu+n-1)\Pi(\alpha+n-1)\Pi(\beta+n-1)\Pi(n+2\nu-1)\,c^n}{2^{n+1}\Pi(n)\Pi(n+\nu)\Pi(\gamma+n-1)\Pi\left(n+\nu-\frac{1}{2}\right)}\,.$$

$$. C_{n-r}^{\mu+n}(y)\, F\left(\frac{\alpha+n}{2},\frac{\alpha+n+1}{2},\frac{\beta+n}{2},\frac{\beta+n+1}{2},n+\nu+1,\frac{\gamma+n}{2},\frac{\gamma+n+1}{2},c^2\right) = \frac{\Pi(\alpha-1)\Pi(\beta-1)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\gamma-1)[\Pi(\nu-1)]^2}\,.$$

$$. \int_{-1}^{+1} F(\alpha,\beta,\gamma,cx)(1-x^2)^{\frac{2\nu-1}{2}}dx\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{C_r^\mu\left(y+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C_r^\mu\left(y+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\} d\varphi$$

$$= \frac{\Pi(\alpha-1)\Pi(\beta-1)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\gamma-1)[\Pi(\nu-1)]^2}\int_{-1}^{+1} F(\alpha,\beta,\gamma,cx)(1-x^2)^{\frac{2\nu-1}{2}}dx\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{C_r^\mu\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)+\right.$$

$$\left.+C_r^\mu\left(y+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\} d\varphi$$

$$\sum_{n=0}^{n=r} \frac{(-2)^n\,\Pi(n+\mu-1)}{\Pi\left(\frac{2n-1}{2}\right)\Pi\left(2n+2\nu-\frac{1}{2}\right)}\, C_{r-n}^{\mu+n}(y)\, C_n^{\nu}(\cos x) = \frac{\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\sin^{2\nu-1}\frac{x}{2}}\,.$$

$$\cdot\int_0^x (\cos\varphi-\cos x)^{\nu-1}\cos\frac{\varphi}{2}\,d\varphi\int_0^{\frac{\pi}{2}}\cos^{4\nu-1}\psi\,\Big\{C_r^{\mu}\Big(y+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\Big)+$$

$$+\,C_r^{\mu}\Big(y+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\Big)\Big\}\,d\psi$$

$$=\frac{\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\mu-1)}{\Pi(\nu-1)\,\Pi(2\nu-1)\sqrt{\pi}\sin^{2\nu-1}\frac{x}{2}}\int_0^x(\cos\varphi-\cos x)^{\nu-1}\cos\frac{\varphi}{2}\,d\varphi\int_0^{\frac{\pi}{2}}\sin^{4\nu-1}\psi\,\Big\{C_r^{\mu}\Big(y+\sin^2\frac{\varphi}{2}\sin^4\psi+$$

$$+\,i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\Big)+C_r^{\mu}\Big(y+\sin^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\Big)\Big\}\,d\psi$$

$$\sum_{n=0}^{n=r}\frac{2^n\,\Pi(n+\mu-1)}{\Pi\left(\frac{2n-1}{2}\right)\Pi\left(2n+2\nu-\frac{1}{2}\right)}\,C_{r-n}^{\mu+n}(y)\,C_n^{\nu}(\cos x)=\frac{\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\cos^{2\nu-1}\frac{x}{2}}\,.$$

$$\cdot\int_x^{\pi}(\cos x-\cos\varphi)^{\nu-1}\sin\frac{\varphi}{2}\,d\varphi\int_0^{\frac{\pi}{2}}\cos^{4\nu-1}\psi\,\Big\{C_r^{\mu}\Big(y+\cos^2\frac{\varphi}{2}\cos^4\psi+i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\Big)+$$

$$+\,C_r^{\mu}\Big(y+\cos^2\frac{\varphi}{2}\cos^4\psi-i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\Big)\Big\}\,d\psi$$

$$=\frac{\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\cos^{2\nu-1}\frac{x}{2}}\int_x^{\pi}(\cos x-\cos\varphi)^{\nu-1}\sin\frac{\varphi}{2}\,d\varphi\int_0^{\frac{\pi}{2}}\sin^{4\nu-1}\psi\,\Big\{C_r^{\mu}\Big(y+\cos^2\frac{\varphi}{2}\sin^4\psi+$$

$$+\,i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\Big)+C_r^{\mu}\Big(y+\cos^2\frac{\varphi}{2}\sin^4\psi-i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\Big)\Big\}\,d\psi$$

$$\sum_{n=0}^{n=r}\frac{(-2)^n\,\Pi(n+\mu-1)}{\Pi\left(\frac{2n+1}{2}\right)\Pi\left(2n+2\nu+\frac{1}{2}\right)}\,C_{r-n}^{\mu+n}(y)\,C_n^{\nu}(\cos x)=\frac{\Pi\left(\frac{2\nu-3}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-2)\,\Pi(2\nu-2)\sin^{2\nu-1}\frac{x}{2}}\,.$$

$$\cdot\int_0^x(\cos\varphi-\cos x)^{\nu-2}\sin\varphi\,d\varphi\int_0^{\frac{\pi}{2}}\cos^{4\nu-3}\psi\,\Big\{\Big(\sin\frac{\varphi}{2}\cos\psi+i\sin\psi\Big)\,C_r^{\mu}\Big(y+\sin^2\frac{\varphi}{2}\cos^4\psi+$$

$$+\,i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\Big)+\Big(\sin\frac{\varphi}{2}\cos\psi+i\sin\psi\Big)C_r^{\mu}\Big(y+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\Big)\Big\}\,d\psi$$

$$=\frac{\Pi\left(\frac{2\nu-3}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-2)\,\Pi(2\nu-2)\sin^{2\nu-1}\frac{x}{2}}\int_0^x(\cos\varphi-\cos x)^{\nu-2}\sin\varphi\,d\varphi\int_0^{\frac{\pi}{2}}\sin^{4\nu-3}\psi\,\Big\{\Big(\sin\frac{\varphi}{2}\sin\psi+i\cos\psi\Big)\,.$$

$$.\,C_r^{\mu}\Big(y+\sin^2\frac{\varphi}{2}\sin^4\psi+i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\Big)+\Big(\sin\frac{\varphi}{2}\sin\psi-i\cos\psi\Big)\,.$$

$$.\,C_r^{\mu}\Big(y+\sin^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\Big)\Big\}\,d\psi$$

$$\sum_{n=0}^{n=r} \frac{2^n \Pi(n+\mu-1)}{\Pi\left(\frac{2n+1}{2}\right)\Pi\left(2n+2\nu+\frac{1}{2}\right)} C_{r-n}^{\mu+n}(y)\, C_n^{\nu}(\cos x) = \frac{\Pi\left(\frac{2\nu-3}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-2)\Pi(2\nu-2)\cos^{2\nu-1}\frac{x}{2}} \,.$$

$$\cdot\int_x^{\pi}(\cos x-\cos\varphi)^{\nu-2}\sin\varphi\, d\varphi\int_0^{\frac{\pi}{2}}\cos^{4\nu-3}\psi\left\{\left(\cos\frac{\varphi}{2}\cos\psi+i\sin\psi\right)C_r^{\mu}\left(y+\cos^2\frac{\varphi}{2}\cos^4\psi+\right.\right.$$

$$\left.\left.+i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)+\left(\cos\frac{\varphi}{2}\cos\psi-i\sin\psi\right)C_r^{\mu}\left(y+\cos^2\frac{\varphi}{2}\cos^4\psi-i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)\right\}d\psi$$

$$=\frac{\Pi\left(\frac{2\nu-3}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-2)\Pi(2\nu-2)\cos^{2\nu-1}\frac{x}{2}}\int_x^{\pi}(\cos x-\cos\varphi)^{\nu-2}\sin\varphi\, d\varphi\int_0^{\frac{\pi}{2}}\sin^{4\nu-3}\psi\left\{\left(\cos\frac{\varphi}{2}\sin\psi+i\cos\psi\right)\right. .$$

$$.\, C_r^{\mu}\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi+i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)+\left(\cos\frac{\varphi}{2}\sin\psi-i\cos\psi\right) .$$

$$.\, C_r^{\mu}\left.\left(y+\cos^2\frac{\varphi}{2}\sin^4\psi-i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)\right\}d\psi .$$

Aus diesen Gleichungen kann man unter Berücksichtigung der Formeln

$$\left[z^{-\rho}J^{\rho}_{(z)}\right]_{z=0}=\frac{1}{2^{\rho}\Pi(\rho)}$$

$$C^{\sigma}_{2m+1}(0)=0,\quad C^{\sigma}_{2m}(0)=(-1)^m\frac{\Pi(m+\sigma-1)}{\Pi(\sigma-1)\Pi(m)}$$

$$C^{\sigma}_{m}(+1)=\frac{\Pi(m+2\sigma-1)}{\Pi(2\sigma-1)\Pi(m)}$$

$$C^{\sigma}_{m}(-1)=(-1)^m\frac{\Pi(m+2\sigma-1)}{\Pi(2\sigma-1)\Pi(m)}$$

die folgenden Relationen ableiten:

$$\int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}}C_n^{\nu}(x)\,dx\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{\left(x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}J^{\mu}\left(\sqrt{x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\right.$$

$$\left.+\left(x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}J^{\mu}\left(\sqrt{x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\}d\varphi=\frac{(-1)^n\sqrt{\pi}\,\Pi(n+2\nu-1)}{2^{3n+\mu+1}\Pi(\mu+n)} .$$

$$.\,\frac{\Pi(\nu-1)}{\Pi(n)\Pi\left(n+\nu-\frac{1}{2}\right)(n+\nu)}\left[\frac{2^{\nu}\Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(2\nu-1)}\right]^2$$

$$\int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}}C_{2n}^{\nu}(x)\,dx\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{\left(x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\mu}J^{\mu}\left(x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+\left(x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\mu} .\right.$$

$$. J^{\mu}\left(x\cos^2\varphi - \frac{i\sin 2\varphi}{2}\right)\Big\} d\varphi = \int_{-1}^{+1} (1-x^2)^{\frac{2\nu-1}{2}} C_{2n}^{\nu}(x)\,dx \int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \Big\{ \left(x\sin^2\varphi + \frac{i\sin 2\varphi}{2}\right)^{-\mu}.$$

$$. J^{\mu}\left(x\sin^2\varphi + \frac{i\sin 2\varphi}{2}\right) + \left(x\sin^2\varphi - \frac{i\sin 2\varphi}{2}\right)^{-\mu} J^{\mu}\left(x\sin^2\varphi - \frac{i\sin 2\varphi}{2}\right)\Big\} d\varphi =$$

$$= \frac{(-1)^n \Pi(2n+2\nu-1)\Pi(\nu-1)}{2^{4n+\mu-1}(2n+\nu)\Pi(2n)\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)\Pi(n+\mu)} \left[\frac{2^{\nu}\Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(2\nu-1)}\right]^2$$

$$\int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}} C_{2n+1}^{\nu}(x)\,dx\int_0^{\frac{\pi}{2}} \cos^{2\nu-\mu-1}\varphi \Big\{(x\cos\varphi+i\sin\varphi)^{1-\mu} J^{\mu}\left(x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+$$

$$(x\cos\varphi - i\sin\varphi)^{1-\mu} J^{\mu}\left(x\cos^2\varphi - \frac{i\sin 2\varphi}{2}\right)\Big\} d\varphi = \int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}} C_{2n+1}^{\nu}(x)\,dx \int_0^{\frac{\pi}{2}} \sin^{2\nu-\mu-1}\varphi\,.$$

$$. \Big\{(x\sin\varphi + i\cos\varphi)^{1-\mu} J^{\mu}\left(x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right) + (x\sin\varphi - i\cos\varphi)^{1-\mu} J^{\mu}\left(x\sin^2\varphi - \frac{i\sin 2\varphi}{2}\right)\Big\} d\varphi =$$

$$= \frac{(-1)^n\Pi(\nu-1)\Pi(2n+2\nu)}{2^{4n+\mu}(2n+\nu+1)\Pi(2n+1)\Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)\Pi(n+\mu)}\left[\frac{2^{\nu}\Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(2\nu-1)}\right]^2.$$

$$\sum_{n=0}^{n=\infty} \frac{(-1)^n\Pi(n+2\nu-1)z^n}{2^{3n}\Pi(n)\Pi\left(n+\nu-\frac{1}{2}\right)\Pi(n+\mu)} = \frac{2^{\mu+1}\nu}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^{\nu}\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 (1-z^2)\int_{-1}^{+1}\int_0^{\frac{\pi}{2}} \frac{(1-x^2)^{\frac{2\nu-1}{2}}}{(1-2zx+z^2)^{\nu+1}}\,.$$

$$\cos^{2\nu-1}\varphi\Big\{\left(x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}} J^{\mu}\left(\sqrt{x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right) + \left(x\cos^2\varphi - \frac{i\sin 2\varphi}{2}\right)^{-\frac{\mu}{2}}.$$

$$J^{\mu}\left(\sqrt{x\cos^2\varphi - \frac{i\sin 2\varphi}{2}}\right)\Big\} dx\,d\varphi$$

und speciell

$$J^{\frac{2\nu-1}{2}}\left(\sqrt{\frac{z}{2}}\right) = \frac{\nu z^{\frac{2\nu-1}{4}}}{2^{\frac{6\nu-3}{4}}\sqrt{\pi}}\left[\frac{\Pi(2\nu-1)}{\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 (1-z^2)\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^{\nu+1}}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\,.$$

$$.\Big\{\left(x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{2\nu-1}{2}} J^{2\nu-1}\left(\sqrt{x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right) + \left(x\cos^2\varphi - \frac{i\sin 2\varphi}{2}\right)^{-\frac{2\nu-1}{2}}.$$

$$. J^{2\nu-1}\left(\sqrt{x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\Big\} d\varphi$$

$$\int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}}\,dx\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\Big\{\left(x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{2\nu-1}{2}}J^{2\nu-1}\left(\sqrt{x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+$$

$$+\left(x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{2\nu-1}{2}}J^{2\nu-1}\left(\sqrt{x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\Big\}d\varphi = \frac{\sqrt{\pi}\,\Pi\left(\frac{2\nu-1}{2}\right)}{\nu[\Pi(2\nu-1)]^2}$$

$$\frac{\pi}{\sqrt{2}}\sin\sqrt{\frac{z}{2}}=\int_{-1}^{+1}\frac{\sqrt{1-x^2}\,dx}{(1-2zx+z^2)^2}\int_0^{\frac{\pi}{2}}\cos\varphi\left\{\left(x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)^{-\frac{1}{2}}J^1\left(\sqrt{x\cos^2\varphi+\frac{i\sin 2\varphi}{2}}\right)+\right.$$

$$\left.+\left(x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)^{-\frac{1}{2}}J^1\left(\sqrt{x\cos^2\varphi-\frac{i\sin 2\varphi}{2}}\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=\left[\frac{r}{2}\right]}(-1)^n\frac{\Pi(\mu+r-2n-1)\,\Pi(\mu+r-n-1)}{\Pi\left(r-2n+\nu-\frac{1}{2}\right)\Pi(\mu+r-2n-1)\,\Pi(n)}\;C^{\mu}_{r-2n}(x)=$$

$$=\frac{\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{C^{\mu}_r\left(x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C^{\mu}_r\left(x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}d\varphi$$

$$=\frac{\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{C^{\mu}_r\left(x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C^{\mu}_r\left(x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}d\varphi$$

$$\int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}}C^{\nu}_{r-2n}(x)\,dx\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{C^{\mu}_r\left(x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C^{\mu}_r\left(x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}d\varphi=$$

$$=\int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}}C^{\nu}_{r-2n}(x)\,dx\int_0^{\frac{\pi}{2}}\left\{C^{\mu}_r\left(x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C^{\mu}_r\left(x\sin^2\varphi-\frac{i\sin^2\varphi}{2}\right)\right\}\sin^{2\nu-1}\varphi\,d\varphi=$$

$$=(-1)^n\frac{2\Pi(\mu+r-n-1)\,\Pi(\mu-2n-1+r)\,\Pi(2\nu-2n+1+r)\,\Pi(\nu-1)\sqrt{\pi}}{\Pi(\mu+r-2n-1)\,\Pi(\mu-1)\,\Pi\left(\nu-2n-\frac{1}{2}+r\right)(\nu-2n+r)}\left[\frac{2^{\nu}\,\Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(2\nu-1)}\right]^2.$$

$$\sum_{n=0}^{n=\left[\frac{r}{2}\right]}\frac{(-1)^n\,\Pi(\mu+r-n-1)}{2^{2n}\,\Pi\left(2r-4n+\nu-\frac{1}{2}\right)\Pi\left(r-2n-\frac{1}{2}\right)}\;C^{\nu}_{2r-4n}(x)=\frac{\Pi(\mu-1)}{2^r\,\Pi(\nu-1)}\;.$$

$$.\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{C^{\mu}_r\left(x^2\cos^4\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+C^{\mu}_r\left(x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}d\varphi$$

$$=\frac{\Pi(\mu-1)}{2^r\,\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{C^{\mu}_r\left(x^2\sin^4\varphi+xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\right.$$

$$\left.+\,C^{\mu}_r\left(x^2\sin^4\varphi-xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=\left[\frac{r}{2}\right]}\frac{(-1)^n\,\Pi(\mu+r-n-1)\,C^{\nu}_{2r-4n+1}(x)}{2^{2n}\,\Pi\left(2r-4n+\nu+\frac{1}{2}\right)\Pi\left(r-2n+\frac{1}{2}\right)}=\frac{\Pi(\mu-1)}{2^r\,\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{(x\cos\varphi+i\sin\varphi)\,.\right.$$

$$\left..\,C^{\mu}_r\left(x^2\cos^4\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+(x\cos\varphi-i\sin\varphi)\,C^{\mu}_r\left(x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}d\varphi$$

$$= \frac{\Pi(\mu-1)}{2^r\,\Pi(\nu-1)}\int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \Big\{ (x\sin\varphi + i\cos\varphi)\, C_r^\mu\Big(x^2\sin^4\varphi + xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\Big) +$$

$$+ (x\sin\varphi - i\cos\varphi)\, C_r^\mu\Big(x^2\sin^4\varphi - xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\Big)\Big\}\, d\varphi$$

$$\int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}} C_{2r-4n}^\nu(x)\,dx \int_0^{\frac{\pi}{2}} \cos^{2\nu-1}\varphi \Big\{ C_r^\mu\Big(x^2\cos^4\varphi + xi\cos^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\Big) +$$

$$+ C_r^\mu\Big(x^2\cos^4\varphi - xi\cos^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\Big)\Big\}\, d\varphi = \int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}} C_{2r-4n}^\nu(x)\,dx \int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi\,.$$

$$\Big\{ C_r^\mu\Big(x^2\sin^4\varphi + xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\Big) + C_r^\mu\Big(x^2\sin^4\varphi - xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\Big)\Big\}\, d\varphi =$$

$$= \frac{(-1)^n\,\Pi(\mu+r-n-1)\,\Pi(\nu-1)\,\Pi(2r-4n+2\nu-1)}{2^{r-2n-1}\,\Pi(\mu-1)\,(2r-4n+\nu)\,\Pi(2r-4n)\,\Pi\Big(2r-4n+\nu-\frac{1}{2}\Big)\,\Pi\Big(r-2n-\frac{1}{2}\Big)}\left[\frac{2^\nu\,\Pi\Big(\frac{2\nu-1}{2}\Big)}{\Pi(2\nu-1)}\right]^2$$

$$\int_{-1}^{+1}(1-x^2)^{\frac{2\nu-1}{2}} C_{2r-4n+1}^\nu(x)\,dx \int_0^{\frac{\pi}{2}} \cos^{2\nu-1}\varphi \Big\{ (x\cos\varphi + i\sin\varphi)\, C_r^\mu\Big(x^2\cos^4\varphi + xi\cos^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\Big) +$$

$$+ (x\cos\varphi - i\sin\varphi)\, C_r^\mu\Big(x^2\cos^4\varphi - xi\cos^2\varphi\sin 2\varphi\, d\varphi - \frac{\sin^2 2\varphi}{4}\Big)\Big\}\, d\varphi = \int_{-1}^{+1}\int_0^{\frac{\pi}{2}} (1-x^2)^{\frac{2\nu-1}{2}} C_{2r-4n+1}^\nu(x)\,.$$

$$.\ \sin^{2\nu-1}\varphi \Big\{ (x\sin\varphi + i\cos\varphi)\, C_r^\mu\Big(x^2\sin^4\varphi + xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\Big) +$$

$$+ (x\sin\varphi - i\cos\varphi)\, C_r^\mu\Big(x^2\sin^4\varphi - xi\sin^2\varphi\sin 2\varphi - \frac{\sin^2 2\varphi}{4}\Big)\Big\}\, dx\, d\varphi =$$

$$= \frac{(-1)^n\,\Pi(\mu+r-n-1)\,\Pi(\nu-1)\,\Pi(2r-4n-2\nu)}{2^{r-2n}\,\Pi(\mu-1)\,(2r-4n+\nu+1)\,\Pi(2r-4n+1)\,\Pi\Big(2r-4n+\nu+\frac{1}{2}\Big)\,\Pi\Big(r-2n+\frac{1}{2}\Big)}\left[\frac{2^\nu\,\Pi\Big(\frac{2\nu-1}{2}\Big)}{\Pi(2\nu-1)}\right]^2.$$

$$\sum_{n=0}^{n=\left[\frac{r}{2}\right]} (-1)^n \frac{\Pi(\mu+r-n-1)\,\Pi(r-2n+2\nu-1)\,z^{r-2n}}{\Pi(r-2n)\,\Pi\Big(r-2n+\nu-\frac{1}{2}\Big)(r-2n+\nu)} = \frac{2\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\Big(\frac{2\nu-1}{2}\Big)}\right]^2.$$

$$.\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^\nu}\int_0^{\frac{\pi}{2}} \cos^{2\nu-1}\varphi \Big\{ C_r^\mu\Big(x\cos^2\varphi + \frac{i\sin 2\varphi}{2}\Big) + C_r^\mu\Big(x\cos^2\varphi - \frac{i\sin 2\varphi}{2}\Big)\Big\}\, d\varphi$$

$$= \frac{2\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\Big(\frac{2\nu-1}{2}\Big)}\right]^2 \int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^\nu}\int_0^{\frac{\pi}{2}} \sin^{2\nu-1}\varphi \Big\{ C_r^\mu\Big(x\cos^2\varphi + \frac{i\sin 2\varphi}{2}\Big) +$$

$$+ C_r^\mu\Big(x\cos^2\varphi - \frac{i\sin 2\varphi}{2}\Big)\Big\}\, d\varphi$$

$$\sum_{n=0}^{n=\left[\frac{r}{2}\right]} (-1)^n \frac{\Pi(\mu+r-n-1)\,\Pi(r-2n+2\nu-1)\,z^{r-2n}}{\Pi(r-2n)\,\Pi\left(r-2n+\nu-\frac{1}{2}\right)} = \frac{2\nu\,\Pi(\mu-1)(1-z^2)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 .$$

$$\cdot\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^{\nu+1}}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{C_r^\mu\left(x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C_r^\mu\left(x\cos^2\varphi-\frac{i\sin^2\varphi}{2}\right)\right\}d\varphi$$

$$=\frac{2\nu\,\Pi(\mu-1)(1-z^2)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^{\nu+1}}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{C_r^\mu\left(x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)+\right.$$

$$\left.+\,C_r^\mu\left(x\sin^2\varphi-\frac{i\sin^2\varphi}{2}\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=\left[\frac{r}{2}\right]}(-1)^n\frac{\Pi(\mu+r-n-1)\,C_n^\nu(\cos x)}{2^{2n}\,\Pi\left(r-2n-\frac{1}{2}\right)\Pi\left(2r-4n+2\nu-\frac{1}{2}\right)}=\frac{(-1)^r\,\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\mu-1)}{2^r\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\sin^{2\nu-1}\frac{x}{2}} .$$

$$\cdot\int_0^x(\cos\varphi-\cos x)^{\nu-1}\cos\frac{\varphi}{2}\,d\varphi\int_0^{\frac{\pi}{2}}\cos^{4\nu-1}\psi\left\{C_r^\mu\left(\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)+\right.$$

$$\left.+\,C_r^\mu\left(\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)\right\}d\psi$$

$$=\frac{(-1)^r\,\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\mu-1)}{2^r\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\sin^{2\nu-1}\frac{x}{2}}\int_0^x(\cos\varphi-\cos x)^{\nu-1}\cos\frac{\varphi}{2}\,d\varphi\int_0^{\frac{\pi}{2}}\sin^{4\nu-1}\psi\ .$$

$$\cdot\left\{C_r^\mu\left(\sin^2\frac{\varphi}{2}\sin^4\psi+i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)+C_r^\mu\left(\sin^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)\right\}d\psi$$

$$\sum_{n=0}^{n=r}\frac{\Pi(\mu+n-1)\,\Pi(r+n+2\mu-1)}{\Pi\left(n+\nu-\frac{1}{2}\right)\Pi(2\mu+2n-1)\,\Pi(r-n)}\,C_n^\nu(x)=\frac{\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\ .$$

$$.\left\{C_r^\mu\left(1+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C_r^\mu\left(1+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}d\varphi$$

$$=\frac{\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{C_r^\mu\left(1+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C_r^\mu\left(1+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=r}\frac{2^n\,\Pi(n+\mu-1)\,\Pi(r+n+2\mu-1)}{\Pi\left(2n+\nu-\frac{1}{2}\right)\Pi\left(\frac{2n-1}{2}\right)\Pi(r-n)\,\Pi(2\mu+2n-1)}\,C_{2n}^\nu(x)=\frac{\Pi(\mu-1)}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\ .$$

$$.\left\{C_r^\mu\left(1+x^2\cos^4\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+C_r^\mu\left(1+x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}d\varphi$$

$$=\frac{\Pi(\mu-1)}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{C_r^\mu\left(1+x^2\sin^4\varphi+xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\right.$$

$$\left.+\,C_r^\mu\left(1+x^2\sin^4\varphi-xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=r}\frac{2^n\,\Pi(n+\mu-1)\,\Pi(r+n+2\mu-1)}{\Pi\left(2n+\nu+\frac{1}{2}\right)\Pi\left(\frac{2n+1}{2}\right)\Pi(r-n)\,\Pi(2\mu+2n-1)}\,C_{2n+1}^\nu(x)=\frac{\Pi(\mu-1)}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\,\{(x\cos\varphi+i\sin\varphi)\,.$$

$$.\,C_r^\mu\left(1+x^2\cos^4\varphi+xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+(x\cos\varphi-i\sin\varphi)\,C_r^\mu\left(1+x^2\cos^4\varphi-xi\cos^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}d\varphi$$

$$=\frac{\Pi(\mu-1)}{\Pi(\nu-1)}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{(x\sin\varphi+i\cos\varphi)\,C_r^\mu\left(1+x^2\sin^4\varphi+xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)+\right.$$

$$\left.+\,(x\sin\varphi-i\cos\varphi)\,C_r^\mu\left(1+x^2\sin^4\varphi-xi\sin^2\varphi\sin 2\varphi-\frac{\sin^2 2\varphi}{4}\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=r}\frac{\Pi(\mu+n-1)\,\Pi(r+n+2\mu-1)\,\Pi(n+2\nu-1)\,z^n}{\Pi(n)\,\Pi\left(n+\nu-\frac{1}{2}\right)\Pi(r-n)\,\Pi(2\mu+2n-1)\,(n+\nu)}=\frac{2\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2.$$

$$.\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^\nu}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{C_r^\mu\left(1+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C_r^\mu\left(1+x\cos^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}d\varphi$$

$$=\frac{2\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^\nu}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{C_r^\mu\left(1+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)+\right.$$

$$\left.+\,C_r^\mu\left(1+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=r}\frac{\Pi(\mu+n-1)\,\Pi(n+r+2\mu-1)\,\Pi(n+2\nu-1)\,z^n}{\Pi(n)\,\Pi\left(n+\nu-\frac{1}{2}\right)\Pi(r-n)\,\Pi(2\mu+2n-1)}=\frac{2\nu\,\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2(1-z^2)\,.$$

$$.\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^{\nu+1}}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{C_r^\mu\left(1+x\cos^2\varphi+\frac{i\sin 2\varphi}{2}\right)+C_r^\mu\left(1+x\cos^2\varphi-\frac{i\sin^2\varphi}{2}\right)\right\}d\varphi$$

$$=\frac{2\nu\,\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2(1-z^2)\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^{\nu+1}}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{C_r^\mu\left(1+x\sin^2\varphi+\frac{i\sin 2\varphi}{2}\right)+\right.$$

$$\left.+\,C_r^\mu\left(1+x\sin^2\varphi-\frac{i\sin 2\varphi}{2}\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=r} \frac{(-2)^n\, \Pi(n+\mu-1)\,\Pi(n+r+2\mu-1)\, C_n^\nu(\cos x)}{\Pi\left(\frac{2n-1}{2}\right)\Pi\left(2n+2\nu-\frac{1}{2}\right)\Pi(r-n)\,\Pi(2\mu+2n-1)} = \frac{\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\sin^{2\nu-1}\frac{x}{2}} \,.$$

$$.\int_0^x (\cos\varphi-\cos x)^{\nu-1}\cos\frac{\varphi}{2}\,d\varphi\int_0^{\frac{\pi}{2}}\cos^{4\nu-1}\psi\left\{C_r^\mu\left(1+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)+\right.$$

$$\left.+\,C_r^\mu\left(1+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)\right\}d\psi$$

$$\sum_{n=0}^{n=r} \frac{2^n\, \Pi(n+\mu-1)\,\Pi(n+r+2\mu-1)\, C_n^\nu(\cos x)}{\Pi\left(\frac{2n-1}{2}\right)\Pi\left(2n+2\nu-\frac{1}{2}\right)\Pi(r-n)\,\Pi(2\mu+2\nu-1)} = \frac{\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\cos^{2\nu-1}\frac{x}{2}} \,.$$

$$.\int_x^\pi (\cos x-\cos\varphi)^{\nu-1}\sin\frac{\varphi}{2}\,d\varphi\int_0^{\frac{\pi}{2}}\cos^{4\nu-1}\psi\left\{C_r^\mu\left(1+\cos^2\frac{\varphi}{2}\cos^4\varphi+i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)+\right.$$

$$\left.+\,C_r^\mu\left(1+\cos^2\frac{\varphi}{2}\cos^4\psi-i\cos\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\varphi}{4}\right)\right\}d\psi$$

$$\sum_{n=0}^{n=r} \frac{(-1)^n\, \Pi(\mu+n-1)\,\Pi(r+n+2\mu-1)\,\Pi(n+2\nu-1)\, z^n}{\Pi(n)\,\Pi\left(n+\nu-\frac{1}{2}\right)\Pi(r-n)\,\Pi(2\mu+2n-1)\,(n+\nu)} = (-1)^r\,\frac{2\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2 .$$

$$.\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^\nu}\int_0^{\frac{\pi}{2}}\cos^{2\nu-1}\varphi\left\{C_r^\mu\left(x\cos^2\varphi+\frac{i\sin 2\varphi}{2}-1\right)+C_r^\mu\left(x\cos^2\varphi-\frac{i\sin 2\varphi}{2}-1\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=r} \frac{(-1)^n\, \Pi(\mu+n-1)\,\Pi(r+n+2\mu-1)\,\Pi(n+2\nu-1)\, z^n}{\Pi(n)\,\Pi\left(n+\nu-\frac{1}{2}\right)\Pi(r-n)\,\Pi(2\mu+2n-1)} = (-1)^r\,\frac{2\nu\,\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)}\left[\frac{\Pi(2\nu-1)}{2^\nu\,\Pi\left(\frac{2\nu-1}{2}\right)}\right]^2(1-z^2)\,.$$

$$.\int_{-1}^{+1}\frac{(1-x^2)^{\frac{2\nu-1}{2}}\,dx}{(1-2zx+z^2)^{\nu+1}}\int_0^{\frac{\pi}{2}}\sin^{2\nu-1}\varphi\left\{C_r^\mu\left(x\sin^2\varphi+\frac{i\sin 2\varphi}{2}-1\right)+C_r^\mu\left(x\sin^2\varphi-\frac{i\sin 2\varphi}{2}-1\right)\right\}d\varphi$$

$$\sum_{n=0}^{n=r} \frac{2^n\, \Pi(n+\mu-1)\,\Pi(n+r+2\mu-1)\, C_n^\nu(\cos x)}{\Pi\left(\frac{2n-1}{2}\right)\Pi\left(2n+2\nu-\frac{1}{2}\right)\Pi(r-n)\,\Pi(2\mu+2n-1)} = (-1)^r\,\frac{\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\sin^{2\nu-1}\frac{x}{2}} \,.$$

$$.\int_0^x (\cos\varphi-\cos x)^{\nu-1}\cos\frac{\varphi}{2}\,d\varphi\int_0^{\frac{\pi}{2}}\cos^{4\nu-1}\psi\left\{C_r^\mu\left(\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}-1\right)+\right.$$

$$\left.+\,C_r^\mu\left(\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}-1\right)\right\}d\psi$$

$$\sum_{n=0}^{n=r} \frac{(-2)^n\, \Pi(n+\mu-1)\,\Pi(n+r+2\mu-1)\, C_n^\nu(\cos x)}{\Pi\left(\frac{2n-1}{2}\right)\Pi\left(2n+2\nu-\frac{1}{2}\right)\Pi(r-n)\,\Pi(2\mu+2n-1)} = (-1)^r \frac{\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-1)\,\Pi(2\nu-1)\cos^{2\nu-1}\frac{x}{2}} \,.$$

$$\cdot\int_x^\pi (\cos x-\cos\varphi)^{\nu-1}\sin\frac{\varphi}{2}\,d\varphi\int_0^{\frac{\pi}{2}}\sin^{4\nu-1}\psi\left\{C_r^\mu\left(\cos^2\frac{\varphi}{2}\sin^4\psi+i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}-1\right)+\right.$$

$$\left.+\,C_r^\mu\left(\cos^2\frac{\varphi}{2}\sin^4\psi-i\cos\frac{\varphi}{2}\sin^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}-1\right)\right\}d\psi$$

$$\sum_{n=0}^{n=r} \frac{(-2)^n\, \Pi(n+\mu-1)\,\Pi(n+r+2\mu-1)\, C_n^\nu(\cos x)}{\Pi\left(\frac{2n+1}{2}\right)\Pi\left(2n+2\nu+\frac{1}{2}\right)\Pi(r-n)\,\Pi(2\mu+2n-1)} = \frac{\Pi\left(\frac{2\nu-3}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-2)\,\Pi(2\nu-2)\sin^{2\nu-1}\frac{x}{2}} \,.$$

$$\cdot\int_0^x (\cos\varphi-\cos x)^{\nu-2}\sin\varphi\, d\varphi\int_0^{\frac{\pi}{2}}\cos^{4\nu-3}\psi\left\{\left(\sin\frac{\varphi}{2}\cos\psi+i\sin\psi\right)\right. .$$

$$.\, C_r^\mu\left(1+\sin^2\frac{\varphi}{2}\cos^4\psi+i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)+$$

$$\left.+\left(\sin\frac{\varphi}{2}\cos\psi-i\sin\psi\right)C_r^\mu\left(1+\sin^2\frac{\varphi}{2}\cos^4\psi-i\sin\frac{\varphi}{2}\cos^2\psi\sin 2\psi-\frac{\sin^2 2\psi}{4}\right)\right\}d\psi$$

$$\sum_{\mu=0}^{\mu=r} \frac{2^n\, \Pi(n+\mu-1)\,\Pi(n+2\mu+r-1)\, C_n^\nu(\cos x)}{\Pi\left(\frac{2n+1}{2}\right)\Pi\left(2n+2\nu+\frac{1}{2}\right)\Pi(r-n)\,\Pi(2\mu+2n-1)} = (-1)^r \frac{\Pi\left(\frac{2\nu-3}{2}\right)\Pi(\mu-1)}{\sqrt{\pi}\,\Pi(\nu-2)\,\Pi(2\nu-2)\sin^{2\nu-1}\frac{x}{2}} \,.$$

$$\cdot\int_0^x (\cos\varphi-\cos x)^{\nu-2}\sin\varphi\, d\varphi\int_0^{\frac{\pi}{2}}\sin^{4\nu-3}\psi\left\{\left(\sin\frac{\varphi}{2}\sin\psi+i\cos\psi\right)\right. .$$

$$.\, C_r^\mu\left(\sin^2\frac{\varphi}{2}\sin^4\psi+i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-1-\frac{\sin^2 2\psi}{4}\right)+$$

$$\left.+\left(\sin\frac{\varphi}{2}\sin\psi-i\cos\psi\right)C_r^\mu\left(\sin^2\frac{\varphi}{2}\sin^4\psi-i\sin\frac{\varphi}{2}\sin^2\psi\sin 2\psi-1-\frac{\sin^2 2\psi}{4}\right)\right\}d\psi.$$

3. Aus der Verbindung der bekannten Relationen

$$(1-x^2)\left[(1-x^2)^{\frac{2\mu-1}{2}}C_n^\mu(x)\right]' = -\frac{(n+1)(n+2\mu-1)}{2(\mu-1)}(1-x^2)^{\frac{2\mu-1}{2}}C_{n+1}^{\mu-1}(x)$$

16a) $$C_{n+1}^{\mu-1}(x) = \frac{2(\mu-1)}{n+1}\left\{x\,C_n^\mu(x)-C_{n-1}^\mu(x)\right\}$$

folgt die Gleichung

17) $$(1-x^2)\left[(1-x^2)^{\frac{2\mu-1}{2}}C_n^\mu(x)\right]' = (n+2\mu-1)(1-x^2)^{\frac{2\mu-1}{2}}\left\{C_{n-1}^\mu(x)-x\,C_n^\mu(x)\right\}$$

während die Vereinigung der Formel 9) mit

$$(1-x^2)\left[C_{n-1}^\mu(x)\right]' = (n+2\mu-2)\,C_{n-2}^\mu(x)-(n-1)\,x\,C_{n-1}^\mu(x)$$

die Beziehung

18) $$(1-x^2)\left[C_{n-1}^\mu(x)\right]' = (n+2\mu-1)\,x\,C_{n-1}^\mu(x)-n\,C_n^\mu(x)$$

liefert. Aus den Gleichungen 17) und 18) ergibt sich sofort die Relation

19) $$(1-x^2)\left[(1-x^2)^{\frac{2\mu-1}{2}} C_n^{\mu}(x)\, C_{n-1}^{\mu}(x)\right]' = (1-x^2)^{\frac{2\mu-1}{2}} \left\{(n+2\mu-1)\,(C_{n-1}^{\mu}(x))^2 - n\,(C_n^{\mu}(x))^2\right\}$$

durch deren Integration zwischen den Grenzen 0 und x die Formel

20) $$\int_0^x \left\{(n+2\mu-1)\,(C_{n-1}^{\mu}(x))^2 - n\,(C_n^{\mu}(x))^2\right\} (1-x^2)^{\frac{2\mu-3}{2}}\, dx = (1-x^2)^{\frac{2\mu-1}{2}}\, C_n^{\mu}(x)\, C_{n-1}^{\mu}(x)$$

entsteht. Aus dieser Gleichung leitet man leicht folgende bemerkenswerte Relationen ab:

$$(1-x^2)^{\frac{2\mu-1}{2}} \sum_{\lambda=1}^{\lambda=n} \frac{C_{\lambda}^{\mu}(x)\, C_{\lambda-1}^{\mu}(x)}{\lambda} = (2\mu-1)\int_0^x (1-x^2)^{\frac{2\mu-3}{2}} \sum_{\lambda=1}^{\lambda=n} \frac{(C_{\lambda-1}^{\mu}(x))^2}{\lambda}\, dx + \int_0^x \left\{1-(C_n^{\mu}(x))^2\right\} (1-x^2)^{\frac{2\mu-3}{2}}\, dx$$

$$(1-x^2)^{\frac{2\mu-1}{2}} \sum_{\lambda=1}^{\lambda=n} \frac{C_{\lambda}^{\mu}(x)\, C_{\lambda-1}^{\mu}(x)}{\lambda+2\mu-1} = (2\mu-1)\int_0^x (1-x^2)^{\frac{2\mu-3}{2}} \sum_{\lambda=1}^{\lambda=n} \frac{(C_{\lambda}^{\mu}(x))^2}{\lambda+2\mu-1}\, dx + \int_0^x \left\{1-(C_n^{\mu}(x))^2\right\} (1-x^2)^{\frac{2\mu-3}{2}}\, dx$$

$$(1-x^2)^{\frac{2\mu-1}{2}} \sum_{\lambda=1}^{\lambda=n} \frac{C_{\lambda}^{\mu}(x)\, C_{\lambda-1}^{\mu}(x)}{\lambda(\lambda+2\mu-1)} = (2\mu-2)\int_0^x (1-x^2)^{\frac{2\mu-3}{2}} \left\{1+\sum_{\lambda=1}^{\lambda=n-1} \frac{(C_{\lambda}^{\mu}(x))^2}{(\lambda+2\mu-1)(\lambda+1)} - \frac{(C_n^{\mu}(x))^2}{n+2\mu-1}\right\} dx \left(\mu \lessgtr \frac{1}{2}\right)$$

$$(1-x^2)^{\frac{2\mu-1}{2}} \sum_{\lambda=1}^{\lambda=n} C_{\lambda}^{\mu}(x)\, C_{\lambda-1}^{\mu}(x) = 2\mu\int_0^x (1-x^2)^{\frac{2\mu-3}{2}} \sum_{\lambda=1}^{\lambda=n} (C_{\lambda-1}^{\mu}(x))^2\, dx - n\int_0^x (C_n^{\mu}(x))^2 (1-x^2)^{\frac{2\mu-3}{2}}\, dx$$

$$(1-x^2)^{\frac{2\mu-1}{2}} \sum_{\lambda=1}^{\lambda=n} (-1)^{\lambda-1} \frac{C_{\lambda}^{\mu}(x)\, C_{\lambda-1}^{\mu}(x)}{\lambda} = (2\mu-1)\int_0^x (1-x^2)^{\frac{2\mu-3}{2}} \sum_{\lambda=1}^{\lambda=n} (-1)^{\lambda-1} \frac{(C_{\lambda-1}^{\mu}(x))^2}{\lambda} +$$

$$\int_0^x \left\{1+2\sum_{\lambda=1}^{\lambda=n-1} (-1)^{\lambda}\,(C_{\lambda}^{\mu}(x))^2 + (-1)^n\,(C_n^{\mu}(x))^2\right\} (1-x^2)^{\frac{2\mu-3}{2}}\, dx.$$

Verbindet man die Gleichung 8) mit 16 a), so erhält man die Formeln:

$$2\mu(1-x)\left\{C_{n-1}^{\mu+1}(x) + C_{n-2}^{\mu+1}(x)\right\} = (n+2\mu-1)\, C_{n-1}^{\mu}(x) - n\, C_n^{\mu}(x)$$

$$2\mu(1+x)\left\{C_{n-1}^{\mu+1}(x) - C_{n-2}^{\mu+1}(x)\right\} = (n+2\mu-1)\, C_{n-1}^{\mu}(x) + n\, C_n^{\mu}(x)$$

welche die Relationen

21) $$4\mu^2(1-x^2)\left\{(C_{n-1}^{\mu+1}(x))^2 - (C_{n-2}^{\mu+1}(x))^2\right\} = (n+2\mu-1)^2\,(C_{n-1}^{\mu}(x))^2 - n^2\,(C_n^{\mu}(x))^2$$

22) $$\frac{d\left((C_n^{\mu}(x))^2 + (C_{n-1}^{\mu}(x))^2\right)}{dx} = 2x\,\frac{d\left(C_n^{\mu}(x)\, C_{n-1}^{\mu}(x)\right)}{dx} + (2\mu-1)\, C_n^{\mu}(x)\, C_{n-1}^{\mu}(x)$$

liefern.

Aus diesen Gleichungen ergeben sich die folgenden Beziehungen:

$$n(1-x^2)^{\frac{2\mu-1}{2}}\, C_{n-1}^{\mu}(x)\, C_n^{\mu}(x) =$$

$$= 4\mu^2\int_0^x (1-x^2)^{\frac{2\mu-1}{2}} \left\{(C_{n-1}^{\mu+1}(x))^2 - (C_{n-2}^{\mu+1}(x))^2\right\} dx + (2\mu-1)(n+2\mu-1)\int_0^x (1-x^2)^{\frac{2\mu-3}{2}}\,(C_{n-1}^{\mu}(x))^2\, dx$$

$$(n+2\mu-1)(1-x^2)^{\frac{2\mu-1}{2}} C_n^{\mu}(x)\, C_{n-1}^{\mu}(x) =$$

$$= 4\mu^2 \int_0^x (1-x^2)^{\frac{2\mu-1}{2}} \{(C_{n-1}^{\mu+1}(x))^2 - (C_{n-2}^{\mu+1}(x))^2\}\, dx + (2\mu-1)\, n \int_0^x (1-x^2)^{\frac{2\mu-3}{2}} (C_n^{\mu}(x))^2\, dx$$

$$2(1-\mu) \sum_{\lambda=0}^{\lambda=n-1} (C_{\lambda}^{\mu}(x))^2 - n\,(C_n^{\mu}(x))^2 =$$

$$= 4\mu^2 (1-x^2) \sum_{\lambda=0}^{\lambda=n-2} \frac{(C_{n-\lambda-2}^{\mu+1}(x))^2}{(n-\lambda)(n-\lambda-1)} + \frac{4\mu^2 (1-x^2)\, (C_{n-1}^{\mu+1}(x))^2}{n} + (2\mu-1) \sum_{\lambda=0}^{\lambda=n-1} \frac{2\mu-1}{(n-\lambda)} (C_{n-\lambda-1}^{\mu}(x))^2$$

$$\int_{-1}^{+1} (1-x^2)^{\lambda}\, C_{n-\lambda}^{\mu+\lambda}(x)\, C_n^{\mu}(x)\, dx = 0 \quad \left(\mu \lessgtr \frac{3}{2}\right)$$

$$\int_0^x x^{\frac{2\mu-3}{2}} \{C_{n-1}^{\mu+1}(x)\, C_n^{\mu}(x) + C_{n-2}^{\mu+1}(x)\, C_{n-1}^{\mu}(x)\}\, dx = \frac{x^{\frac{2\mu-1}{2}}}{2\mu} C_n^{\mu}(x)\, C_{n-1}^{\mu}(x) \left(\mu \geqq \frac{1}{2}\right)$$

$$\sum_{\lambda=1}^{\lambda=n} (-1)^{\lambda-1} C_{\lambda}^{\mu}(x)\, C_{\lambda-1}^{\mu}(x) = (-1)^{n-1} \frac{2\mu}{x^{\frac{2\mu-1}{2}}} \int_0^x x^{\frac{2\mu-3}{2}} C_n^{\mu}(x)\, C_{n-1}^{\mu+1}(x)\, dx$$

$$\sum_{\lambda=1}^{\lambda=n} (-1)^{\lambda-1} \frac{\Pi(\lambda+2\mu-1)\, \Pi(\lambda+2\mu-2)}{\Pi(\lambda)\, \Pi(\lambda-1)\, [\Pi(2\mu-1)]^2} = (-1)^{n-1} 2\mu \int_0^1 x^{\frac{2\mu-3}{2}} C_n^{\mu}(x)\, C_{n-1}^{\mu+1}(x)\, dx$$

und speciell

$$\int_0^1 \frac{P_n(x)\, d\, P_n(x)}{x} = 0 \quad (n \text{ ungerade})$$

$$\int_0^1 \frac{P_n(x)\, d\, P_n(x)}{x} = 1 \quad (n \text{ gerade})$$

$$(C_n^{\mu}(x))^2 + (C_{n-1}^{\mu}(x))^2 - 2x\, C_n^{\mu}(x)\, C_{n-1}^{\mu}(x) = (2\mu-3) \int_0^x C_n^{\mu}(x)\, C_{n-1}^{\mu}(x)\, dx + \left[\frac{\Pi\left(\left[\frac{n}{2}\right]+\mu-1\right)}{\Pi(\mu-1)\, \Pi\left(\left[\frac{n}{2}\right]\right)}\right]^2$$

$$= (2\mu-3) \int_{-1}^x C_n^{\mu}(x)\, C_{n-1}^{\mu}(x)\, dx + \left[\frac{\Pi(n+2\mu-2)}{\Pi(n)\, \Pi(2\mu-2)}\right]^2.$$

Setzt man in dieser Formel $x = \cos\varphi$ und beachtet, dass alsdann die linke Seite derselben das Quadrat der dritten Seite eines veränderlichen Dreieckes ist, dessen zwei andere Seiten die Längen $C_n^{\mu}(\cos\varphi)$ und $C_{n-1}^{\mu}(\cos\varphi)$ besitzen und den Winkel φ einschliessen, während der auf der rechten Seite unter dem Integralzeichen stehende Ausdruck die doppelte Fläche dieses Dreieckes vorstellt, so erhält man einen interessanten geometrischen Lehrsatz.

Multiplicirt man die von mir in diesen Denkschriften abgeleitete Gleichung

$$C_{n-\lambda}^{\nu}(x) = \frac{\Pi(n+2\nu-\lambda-1)\, x^{n-\lambda-1}}{2^{2\nu-2}\, \Pi(n-\lambda)\, [\Pi(\nu-1)]^2\, (1-x^2)^{\frac{2\nu-1}{2}}} \int_0^{\arccos x} \frac{(\cos^2\psi - x^2)^{\nu-1} \cos(n-\lambda)\psi\, d\psi}{\cos^{n-\lambda+2\nu}\psi}$$

mit $\frac{(-x)^\lambda}{\Pi(\lambda)\,\Pi(n+2\nu-\lambda-1)}$ und summirt bezüglich λ von 0 bis n, so erhält man

$$\sum_{\lambda=0}^{\lambda=n}(-1)^\lambda \frac{x^\lambda C^\nu_{n-\lambda}(x)}{\Pi(\lambda)\,\Pi(n+2\nu-\lambda-1)} = \frac{x^{n+1}}{2^{2\nu-2}\,[\Pi(\nu-1)]^2\,\Pi(n)\,(1-x^2)^{\frac{2\nu-1}{2}}}\int_0^{\arccos x}\frac{(\cos^2\psi-x^2)^{\nu-1}}{\cos^{n+2\nu}\psi}\,d\psi\,.$$

$$.\sum_{\lambda=0}^{\lambda=n}(-1)^\lambda\binom{n}{\lambda}\cos^\lambda\psi\cos(n-\lambda)\psi$$

oder weil

$$\sum_{\lambda=0}^{\lambda=n}(-1)^\lambda\binom{n}{\lambda}\cos^\lambda\psi\cos(n-\lambda)\psi$$

der reelle Bestandtheil von $(e^{\psi i}-\cos\psi)^n = (i\sin\psi)^n$ ist,

$$\sum_{\lambda=0}^{\lambda=2r+1}(-1)^\lambda\frac{x^\lambda C^\nu_{2r-\lambda+1}(x)}{\Pi(\lambda)\,\Pi(2r+2\nu-\lambda)} = 0$$

$$\sum_{\lambda=0}^{\lambda=2r}(-1)^\lambda\frac{x^\lambda C^\nu_{2r-\lambda}(x)}{\Pi(\lambda)\,\Pi(2r+2\nu-\lambda-1)} = \frac{(-1)^r\,x^{2r+1}}{2^{2\nu-2}\,[\Pi(\nu-1)]^2\,\Pi(2r)\,(1-x^2)^{\frac{2\nu-1}{2}}}\int_0^{\arccos x}\frac{(\cos^2\psi-x^2)^{\nu-1}\sin^{2r}\psi\,d\psi}{\cos^{2r+2\nu}\psi}\,.$$

Transformirt man das auf der rechten Seite dieser Gleichung stehende Integral durch die Substitution

$$\sin\psi = z\sqrt{1-x^2}$$

so verwandelt sich dasselbe in

$$(1-x^2)^{r+\nu-\frac{1}{2}}\int_0^1(1-z^2)^{\nu-1}z^{2r}\left(1-(1-x^2)z^2\right)^{-\left(r+\nu+\frac{1}{2}\right)}dz = \frac{\Pi\left(r-\frac{1}{2}\right)\Pi(\nu-1)}{2\,\Pi\left(r+\nu-\frac{1}{2}\right)x^{2r+1}}(1-x^2)^{r+\nu-\frac{1}{2}}$$

und demnach geht die letzte Gleichung in die folgende über

$$\sum_{\lambda=0}^{\lambda=2r}(-1)^\lambda\frac{x^\lambda C^\nu_{2r-\lambda}(x)}{\Pi(\lambda)\,\Pi(2r+2\nu-\lambda-1)} = \frac{(-1)^r\sqrt{\pi}\,\Pi(\nu-1)\,(1-x^2)^r}{2^{2r+2\nu-2}\,\Pi(r)\,\Pi\left(r+\nu-\frac{1}{2}\right)[\Pi(\nu-1)]^2}\,.$$

Multiplicirt man die aus 8) und 16*a*) folgende Relation

$$2\nu\,C^\nu_n(x) = \frac{d\,C^\nu_{n-1}(x)}{dx} - 2x\frac{d\,C^\nu_n(x)}{dx} + \frac{d\,C^\nu_{n+1}(x)}{dx}$$

mit $(1-x)^{\nu-1}$ so verwandelt sich dieselbe in die Formel

$$\frac{d\,\{(1-x)^\nu C^\nu_n(x)\}}{dx} = -\frac{(1-x)^{\nu-1}}{2}\left\{\frac{d\,C^\nu_{n-1}(x)}{dx} - 2\frac{d\,C^\nu_n(x)}{dx} + \frac{d\,C^\nu_{n+1}(x)}{dx}\right\}$$

durch deren Integration die Relation

$$\int_1^x(1-x)^{\nu-1}\left\{\frac{d\,C^\nu_{n-1}(x)}{dx} - 2\frac{d\,C^\nu_n(x)}{dx} + \frac{d\,C^\nu_{n+1}(x)}{dx}\right\}dx = -2\,(1-x)^\nu\,C^\nu_n(x)\;(\nu>0)$$

oder auch

$$\nu \int_1^x (1-x)^{\nu-1} \left\{ C_{n-2}^{\nu+1}(x) - 2\, C_{n-1}^{\nu+1}(x) + C_n^{\nu+1}(x) \right\} dx = -(1-x)^\nu\, C_n^\nu(x)$$

entsteht. Schreibt man in derselben für n der Reihe nach n, $n-1$, $n-2, \ldots, 2, 1$ und addirt die dadurch entstehenden Gleichungen, so erhält man die bemerkenswerthe Formel

$$\int_1^x (1-x)^{\nu-1} \left\{ \frac{d\, C_n^\nu(x)}{dx} - \frac{d\, C_{n+1}^\nu(x)}{dx} \right\} dx = 2\nu \int_1^x (1-x)^{\nu-1} \left\{ C_{n-1}^{\nu+1}(x) - C_n^{\nu+1}(x) \right\} dx = 2(1-x)^\nu \sum_{\lambda=0}^{\lambda=n} C_\lambda^\nu(x).$$

Aus der Gleichung $3a$) leitet man leicht die folgende Formel her

$$\sum_{n=0}^{n=\infty} \frac{\Pi(n)}{\Pi(n+2\nu)} \left\{ (n+2\nu)\, x\, C_n^\nu(x) - (n+1)\, C_{n+1}^\nu(x) \right\} = \frac{i\sqrt{1+x}}{2^{2\nu-1}[\Pi(\nu-1)]^2} \int_0^\pi \frac{\sin^{2\nu-1}\varphi \cos\varphi\, d\varphi}{\sqrt{1-x} + i\sqrt{1+x}\cos\varphi}$$

welche sich mit Hilfe von $1a$) sofort in

$$\sum_{n=0}^{n=\infty} \frac{\Pi(n)}{\Pi(n+2\nu)} \left\{ (n+2\nu)\, x\, C_n^\nu(x) - (n+1)\, C_{n+1}^\nu(x) \right\} = \frac{1+x}{\Pi(2\nu)\,(\nu+1)\,(1-x)} \left[\frac{\Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(\nu-1)} \right]^2 F\left(1, \frac{3}{2}, \nu+2, -\frac{1+x}{1-x}\right)$$

überführen lässt.

Aus derselben Gleichung folgt auch die Relation

$$\sum_{n=0}^{n=\infty} \frac{C_n^\nu(x)}{\Pi(n+2\nu-1)} = \frac{\sqrt{\pi}\, e\, J^{\frac{2\nu-1}{2}}\left(\sqrt{1-x^2}\right)}{2^{\nu-\frac{1}{2}}\, \Pi(\nu-1)\, (1-x^2)^{\frac{2\nu-1}{4}}}.$$

Aus den von mir in diesen Denkschriften früher aufgestellten Gleichungen

$$C_n^{\nu_1+\nu_2+\ldots+\nu_r}(x) = \sum_{n_1, n_2, \ldots, n_r} C_{n_1}^{\nu_1}(x)\, C_{n_2}^{\nu_2}(x) \ldots C_{n_r}^{\nu_r}(x) \qquad (n_1 + n_2 + \ldots + n_r = n)$$

$$\frac{1-xz}{(1-2xz+z^2)^\nu} = \sum_{n=0}^{n=\infty} \frac{n+2\nu-2}{2\nu-2}\, C_n^{\nu-1}(x)\, z^n$$

folgt die Beziehung

23) $$\frac{n+2\nu-2}{2\nu-2}\, C_n^{\nu-1}(x) = \sum_{n_1, n_2, \ldots, n_r} C_{n_1}^{\nu_1}(x)\, C_{n_2}^{\nu_2}(x) \ldots C_{n_r}^{\nu_r}(x) - x \sum_{m_1, m_2, \ldots, m_s} C_{m_1}^{\mu_1}(x)\, C_{m_2}^{\mu_2}(x) \ldots C_{m_s}^{\mu_s}(x)$$

$$(n_1 + n_2 + \ldots + n_r = m_1 + m_2 + \ldots + m_s + 1 = n;\ \nu_1 + \nu_2 + \ldots + \nu_r = \mu_1 + \mu_2 + \ldots + \mu_s = \nu,$$
$$0 \leqq n_k \leqq n,\ 0 < m_k + 1 \leqq n)$$

aus welcher sich mit Hilfe der Formel 18) die Relation

24) $$\frac{n+2\nu-2}{2\nu-2}\, C_n^{\nu-1}(x) =$$

$$= (x^2-1)^{\frac{3-2\nu_1}{2}} \sum_{n_2, n_3, \ldots, n_r = 0}^{n_2, n_3, \ldots, n_r = n-1} \frac{C_{n_2}^{\nu_2}(x)\, C_{n_3}^{\nu_3}(x) \ldots C_{n_r}^{\nu_r}(x) \left[(x^2-1)^{\frac{2\nu_1-1}{2}}\, C_{n-n_2-n_3-\ldots-n_r-1}^{\nu_1} \right]'}{n-n_2-n_3-\ldots-n_r} + \sum_{\lambda=2}^{\lambda=r} C_n^{\nu_\lambda}(x)$$

$$= \frac{1}{2\nu_1-2} \sum_{n_2, n_3, \ldots, n_r}^{n_2, n_3, \ldots, n_r = n} (n-n_2-n_3-\ldots-n_r+2\nu_1-2)\, C^{\nu_1}_{n-n_2-n_3-\ldots-n_r}(x)\, C^{\nu_2}_{n_2}(x)\, C^{\nu_3}_{n_3}(x) \ldots C^{\nu_r}_{n_r}(x)$$

$$(\nu_1+\nu_2+\ldots+\nu_r = \nu,\ n-n_2-\ldots-n_r \geqq 1 \text{ bez. } 0)$$

ergibt, aus der sich für $r = 2$ die specielle Beziehung

$$\frac{n+2\nu-2}{2\nu-2} C_n^{\nu-1}(x) = \frac{1}{2\nu-2\nu_1-2} \sum_{\lambda=0}^{\lambda=n-1} (n-\lambda+2\nu-2\nu_1-2)\, C^{\nu_1}_{\lambda}(x)\, C^{\nu-\nu_1-1}_{n-\lambda}(x)$$

ergibt, welche ich schon früher mitgetheilt habe.

Ist in der Gleichung 23) $r = s$, $\nu_k = \mu_k$ und wird sodann für n der Reihe nach $n, n-1, n-2, \ldots, 3, 2, 1$ geschrieben, die λ^{te} von den so entstehenden Gleichungen mit x^λ multiplicirt und die Summe derselben gebildet, so entsteht die Relation

24) $$\sum_{n_1, n_2, \ldots, n_r} C^{\nu_1}_{n_1}(x)\, C^{\nu_2}_{n_2}(x) \ldots C^{\nu_r}_{n_r}(x) = \frac{1}{2\nu-2} \sum_{\lambda=0}^{\lambda=n} (n-\lambda+2\nu-2)\, x^\lambda\, C^{\nu-1}_{n-\lambda}(x)$$

$$(n_1+n_2+\ldots+n_r = n;\ \nu_1+\nu_2+\ldots+\nu_r = \nu).$$

Setzt man nun in dieser Gleichung speciell

$$r = 2,\ \nu_1 = \nu_2$$

multiplicirt mit $(1-x^2)^{\frac{2\nu-1}{2}} dx$, integrirt von $x = -1$ bis $x = +1$ und beachtet, dass für ein ungerades n sämmtliche auf der linken Seite der so entstehenden Gleichung befindlichen Integrale gleich 0 sind, während für ein gerades n nur das mittlere einen von 0 verschiedenen Werth besitzt, so erhält man die Relationen

25) $$\sum_{\lambda=0}^{\lambda=2n} (2n-\lambda+4\nu-2) \int_{-1}^{+1} x^\lambda (1-x^2)^{\frac{2\nu-1}{2}} C^{2\nu-1}_{2n-\lambda}(x)\, dx = \frac{2^{2\nu}(2\nu-1)\Pi(n+2\nu-1)}{(n+\nu)\Pi(n)} \left[\frac{\Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(2\nu-1)}\right]^2$$

$$\sum_{\lambda=0}^{\lambda=2n+1} (2n-\lambda+4\nu-1) \int_{-1}^{+1} x^\lambda (1-x^2)^{\frac{2\nu-1}{2}} C^{2\nu-1}_{2n-\lambda+1}(x)\, dx = 0$$

durch deren Vereinigung mit 24) die bemerkenswerthen Formeln

$$\sum_{n_1, n_2, \ldots, n_r} \int_{-1}^{+1} C^{\nu_1}_{n_1}(x)\, C^{\nu_2}_{n_2}(x) \ldots C^{\nu_r}_{n_r}(x)\, (1-x^2)^{\frac{\nu-1}{2}} dx = 0 \quad (n_1+n_2+\ldots+n_r = 2n+1;\ \nu_1+\nu_2+\ldots+\nu_r = \nu)$$

$$\sum_{n_1, n_2, \ldots, n_r} \int_{-1}^{+1} C^{\nu_1}_{n_1}(x)\, C^{\nu_2}_{n_2}(x) \ldots C^{\nu_r}_{n_r}(x)\, (1-x^2)^{\frac{\nu-1}{2}} dx = \frac{2^{\nu+1}(\nu-1)\Pi(n+\nu-1)}{(2n+\nu)\Pi(n)} \left[\frac{\Pi\left(\frac{\nu-1}{2}\right)}{\Pi(\nu-1)}\right]^2$$

$$(n_1+n_2+\ldots+n_r = 2n;\ \nu_1+\nu_2+\ldots+\nu_r = \nu)$$

entstehen.

Aus der ersten von den Gleichungen 25) ergibt sich für die Coëfficienten $A^{(n)}_{\nu,\mu}$ der Functionen

$$C_n^\nu(x) = \sum_{\mu=0}^{\mu=\left[\frac{n}{2}\right]} A^{(n)}_{\nu,\mu}\, x^{n-2\mu}$$

die Relation

$$\sum_{\lambda=0}^{\lambda=2n}\sum_{\mu=0}^{\mu=n+\left[-\frac{\lambda}{2}\right]}\frac{\Pi\left(n-\mu-\frac{1}{2}\right)}{\Pi(n-\mu+\nu)}(2n-\lambda+4\nu-2)\,A_{2\nu-1,\mu}^{(2n-\lambda)}=\frac{2^{2\nu+1}(2\nu-1)\,\Pi(n+2\nu-1)\,\Pi\left(\frac{2\nu-1}{2}\right)}{(n+\nu)\,\Pi(n)\,[\Pi(2\nu-1)]^2}.$$

Multiplicirt man die von mir abgeleitete Formel

$$C_n^\nu(\cos x)=\sum_{\lambda=0}^{\lambda=n}\frac{\Pi(\nu+\lambda-1)\,\Pi(\nu-\lambda+n-1)}{[\Pi(\nu-1)]^2\,\Pi(\lambda)\,\Pi(n-\lambda)}\cos(n-2\lambda)\,x$$

mit $\cos(n-2\mu)\,x\,dx$ und integrirt von $x=0$ bis $x=\pi$, so erhält man die Relation

$$\frac{2}{\pi}\int_0^\pi C_n^\nu(\cos x)\cos(n-2\mu)\,x\,dx=\frac{\Pi(\nu+\mu-1)\,\Pi(\nu-\mu+n-1)}{[\Pi(\nu-1)]^2\,\Pi(\mu)\,\Pi(n-\mu)}$$

aus welcher sich bei ungeradem n unter Berücksichtigung der bekannten Formel

$$\sum_{\lambda=0}^{\lambda=\left[\frac{n}{2}\right]}\cos(n-2\lambda)\,x\cos(n-2\lambda)\,y=\frac{1}{4}\left\{\frac{\sin(n+1)(x+y)}{\sin(x+y)}+\frac{\sin(n+1)(x-y)}{\sin(x-y)}\right\}$$

für die Functionen $C_n^\nu(\cos x)$ folgender Integralausdruck ergibt

$$C_n^\nu(\cos x)=\frac{1}{2\pi}\int_0^\pi C_n^\nu(\cos y)\left\{\frac{\sin(n+1)(x+y)}{\sin(x+y)}+\frac{\sin(n+1)(x-y)}{\sin(x-y)}\right\}dy.$$

Herr E. Heine hat in seinem Handbuche der Kugelfunctionen[1] mit Hilfe der in einem von Euler am 4. December 1751 an Goldbach geschriebenen Briefe enthaltenen Bemerkung, dass in der Entwicklung von $\sqrt[n]{1-an^2}$ nach aufsteigenden Potenzen von a alle Coëfficienten ganze Zahlen sind, gezeigt, dass das Product $c^{2n}\,C_n^\nu(x)$ nur ganze Coëfficienten besitzt, wenn ν eine rationale Zahl mit dem Nenner c ist. Dieser Satz lässt sich durch einen anderen etwas weiter gehenden ersetzen. Setzt man in der oben benützten Definitionsgleichung der Function $C_n^\mu(x)$ $\mu=\frac{r}{c}$, so nimmt sie die folgende Form an:

$$C_n^{\frac{r}{c}}(x)=\sum_{\lambda=0}^{\lambda=\left[\frac{n}{2}\right]}(-1)^\lambda\frac{\{r+c\,(n-\lambda-1)\}\,\{r+c\,(n-\lambda-2)\}\ldots r}{c^{n-\lambda}\,\Pi(\lambda)\,\Pi(n-2\lambda)}(2x)^{n-2\lambda}.$$

Nun ist aber bekanntlich[2]

$$\frac{m(m+n_1)(m+2n_1)\ldots(m+(i-1)n_1)\,n_1^{\,i-1}}{\Pi(i)}$$

eine ganze Zahl und daher ist der Coëfficient von $x^{n-2\lambda}$ in der auf der rechten Seite dieser Gleichung stehenden Summe das Product aus einer ganzen Zahl und dem Ausdrucke

$$\frac{(n-2\lambda+1)(n-2\lambda+2)\ldots(n-\lambda+1)\,2^{n-2\lambda}}{\Pi(\lambda)\,c^{2(n-\lambda)-1}}.$$

[1] 1. Band, S. 14.

[2] Dieser Satz ist meines Wissens zuerst von Herrn Charles Hermite in der dritten Ausgabe seines Cours d'analyse S. 175 aus dem Eisenstein'schen Satze über die Coëfficienten von Reihen, welche algebraischen Differentialgleichungen genügen, hergeleitet worden. Der Hermite'sche Satz ist übrigens ein ganz specieller Fall eines allgemeinen arithmetischen Theorems, welches ich vor einer Reihe von Jahren meinen Hörern in der von mir geleiteten Abtheilung des mathematischen Seminars an der Innsbrucker Universität mitgetheilt habe und von welchem demnächst Herr J. A. Gmeiner in den Monatsheften der Mathematik und Physik einen Beweis veröffentlichen wird.

Da aber das Product von irgend welchen aufeinander folgenden λ ganzen Zahlen durch das Product der ersten λ ganzen Zahlen theilbar ist, so wird der erwähnte Coëfficient mit $\frac{c^{2(n-\lambda)-1}}{2^{n-2\lambda}}$ multiplicirt ganz und demnach ergibt sich der Satz:

Ist ν eine rationale Zahl mit dem Nenner $2r+1$, so sind sämmtliche Coëfficienten des nach Potenzen von x geordneten Productes $(2r+1)^{2n-1}\, C_n^{\nu}(x)$ ganz, ist aber der Nenner von ν gleich $2^{\mu}(2r+1)$, so besitzt schon das Product $\frac{(2^{\mu}(2r+1))^{2n-1}}{2^n}\, C_n^{\nu}(x)$ nur ganze Coëfficienten.

Setzt man in der von mir in diesen Denkschriften aufgestellten Gleichung

$$C_n^{\nu}(x) = \frac{2^n\, \Pi(n+\nu-1)\, \Pi(n+2\nu-1) \left[\Pi\left(\frac{2n+2\nu-1}{2}\right)\right]^2}{\Pi(\nu-1)\, \Pi(2n+2\nu-1)} \sum_{\lambda=0}^{\lambda=n} \frac{(x-1)^{n-\lambda}(x+1)^{\lambda}}{\Pi\left(\frac{2n+2\nu-2\lambda-1}{2}\right) \Pi\left(\frac{2\nu+2\lambda-1}{2}\right) \Pi(\lambda)\, \Pi(n-\lambda)}$$

der Reihe nach $x = 0, \frac{1}{2}$, so entstehen die Relationen

$$\sum_{\lambda=0}^{\lambda=2r+1} \frac{(-1)^{\lambda}}{\Pi\left(2r+\nu-\lambda+\frac{1}{2}\right) \Pi\left(\nu+\lambda-\frac{1}{2}\right) \Pi(\lambda)\, \Pi(2r-\lambda+1)} = 0$$

$$\sum_{\lambda=0}^{\lambda=2r} \frac{(-1)^{\lambda}}{\Pi\left(2r+\nu-\lambda-\frac{1}{2}\right) \Pi\left(\nu+\lambda-\frac{1}{2}\right) \Pi(\lambda)\, \Pi(2r-\lambda)} = \frac{(-1)^r\, \Pi(4r+2\nu-1)\, \Pi(r+\nu-1)}{2^{2r}\, \Pi(2r+\nu-1)\, \Pi(2r+2\nu-1)\, \Pi(r) \left[\Pi\left(2r+\nu-\frac{1}{2}\right)\right]^2}$$

$$C_n^{\nu}\left(\frac{1}{2}\right) = \frac{\Pi(n+\nu-1)\, \Pi(n+2\nu-1) \left[\Pi\left(n+\nu-\frac{1}{2}\right)\right]^2}{\Pi(\nu-1)\, \Pi(2n+2\nu-1)} \sum_{\lambda=0}^{\lambda=n} (-1)^{n-\lambda} \frac{3^{\lambda}}{\Pi\left(n+\nu-\lambda-\frac{1}{2}\right) \Pi\left(\nu+\lambda-\frac{1}{2}\right) \Pi(\lambda)\, \Pi(n-\lambda)}$$

deren letzte für $\nu = \frac{1}{2}$ die interessante Catalan'sche Formel

$$2^{2n}\, P_n\left(\frac{1}{2}\right) = \sum_{\lambda=0}^{\lambda=n} (-1)^{n-\lambda} \binom{n}{\lambda}^2 3^{\lambda}$$

liefert.

Verbindet man diese letzte Gleichung mit der aus meiner Formel

$$C_n^{\nu}(x) = \frac{\Pi(n+2\nu-1)\sqrt{\pi}}{2^{2\nu-1}\, \Pi(\nu-1)} \sum_{\lambda=0}^{\lambda=\left[\frac{n}{2}\right]} \frac{x^{n-2\lambda}(x^2-1)^{\lambda}}{2^{2\lambda}\, \Pi(\lambda)\, \Pi\left(\nu+\lambda-\frac{1}{2}\right) \Pi(n-2\lambda)}$$

folgenden Relation

$$C_n^{\nu}\left(\frac{1}{2}\right) = \frac{\Pi(n+2\nu-1)\sqrt{\pi}}{2^{n+2\nu-1}\, \Pi(\nu-1)} \sum_{\lambda=0}^{\lambda=\left[\frac{n}{2}\right]} \frac{(-3)^{\lambda}}{2^{2\lambda}\, \Pi(\lambda)\, \Pi\left(\nu+\lambda-\frac{1}{2}\right) \Pi(n-2\lambda)}$$

so entsteht die Beziehung

$$\sum_{\lambda=0}^{\lambda=\left[\frac{n}{2}\right]} \frac{(-3)^\lambda}{2^{2\lambda}\,\Pi(\lambda)\,\Pi\left(\nu+\lambda-\frac{1}{2}\right)\Pi(n-2\lambda)} = \frac{(-1)^n\,\Pi(n+\nu-1)\left[\Pi\left(n+\nu-\frac{1}{2}\right)\right]^2 2^{n+2\nu-1}}{\Pi(2n+2\nu-1)\sqrt{\pi}}\,.$$

$$.\sum_{\lambda=0}^{\lambda=n} \frac{(-3)^\lambda}{\Pi\left(n+\nu-\lambda-\frac{1}{2}\right)\Pi\left(\nu+\lambda-\frac{1}{2}\right)\Pi(\lambda)\,\Pi(n-\lambda)}\,.$$

Die Vergleichung der Coëfficienten von x^n in den beiden eben benützten Entwicklungen von $C_n^\nu(x)$ liefert ferner die Relation

$$\sum_{\lambda=0}^{\lambda=\left[\frac{n}{2}\right]} \frac{1}{2^{2\lambda}\,\Pi(\lambda)\,\Pi\left(\nu+\lambda-\frac{1}{2}\right)\Pi(n-2\lambda)} =$$

$$= \frac{2^{n+2\nu-1}\,\Pi(n+\nu-1)\left[\Pi\left(n+\nu-\frac{1}{2}\right)\right]^2}{\Pi(2n+2\nu-1)\sqrt{\pi}} \sum_{\lambda=0}^{\lambda=n} \frac{1}{\Pi\left(n+\nu-\lambda-\frac{1}{2}\right)\Pi\left(\nu+\lambda-\frac{1}{2}\right)\Pi(n-\lambda)\,\Pi(\lambda)}\,.$$

Setzt man in der von mir aufgestellten Gleichung [1])

$$x^n = \frac{\Pi(n)\,\Pi(\nu-1)}{2^n} \sum_{\lambda=0}^{\lambda=\left[\frac{n}{2}\right]} \frac{n+\nu-2\lambda}{\Pi(\lambda)\,\Pi(n+\nu-\lambda)}\, C_{n-2\lambda}^\nu(x)$$

$x = 1, 0$, so ergeben sich die interessanten Formeln

$$\sum_{\lambda=0}^{\lambda=\left[\frac{n}{2}\right]} \frac{(n+\nu-2\lambda)\,\Pi(n-2\lambda+2\nu-1)}{\Pi(\lambda)\,\Pi(n+\nu-\lambda)\,\Pi(n-2\lambda)} = \frac{2^n\,\Pi(2\nu-1)}{\Pi(n)\,\Pi(\nu-1)}$$

$$\sum_{\lambda=0}^{\lambda=r} (-1)^\lambda \frac{(2r+\nu-2\lambda)\,\Pi(r+\nu-\lambda-1)}{\Pi(\lambda)\,\Pi(2r+\nu-\lambda)\,\Pi(r-\lambda)} = 0.$$

Herr Hofrath Professor Dr. A. Winckler hat in seiner Abhandlung „Über ein Kriterium des Grössten und Kleinsten in der Variationsrechnung" [2]) durch Vergleichung seiner Form der zweiten Variation mit der Legendre-Jacobi'schen zwei bemerkenswerthe Theoreme über das Vorzeichen von gewissen bestimmten Integralen gefunden. Man kann leicht eine Reihe von Sätzen derselben Kategorie, in denen die Functionen $C_n^\nu(x)$ eine Rolle spielen, aufstellen, und von diesen möge der folgende angegeben werden:

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[1] „Über die Functionen $C_n^\nu(x)$ und $D_n^\nu(x)$". Programm der n. ö. Landes-Oberrealschule in Krems vom Jahre 1873.

[2] Sitzungsberichte der k. Akademie der Wissenschaften, mathem.-naturw. Classe, 97. Band, Abtheilung II a, S. 1065—1082. Ein weiteres Analogon der Winckler'schen Theoreme ist der folgende Satz:

Ist $\omega(x)$ eine von $ce^{\frac{\sqrt{nx}}{2}}$ verschiedene differentiirbare Function, welche nebst ihrer stetigen Ableitung im reellen Intervalle $x_0 \ldots x_1$ endlich bleibt, so ist für negative Wertepaare x_0, x_1

$$(-1)^m \int_{x_0}^{x_1} \left\{ x\left(\frac{d\omega(x)}{dx}\right)^2 - n\,\omega^2(x) \right\} e^{-x}\, x^m\, dx > 0 \qquad (m \text{ und } n \text{ ganzzahlig, nicht negativ})$$

Ist $\omega(x)$ eine von $c\left(\frac{x+1}{x-1}\right)^{\pm\frac{n(n+2\nu)}{2}}$ verschiedene differentiirbare Function, welche nebst ihrer stetigen ersten Ableitung im reellen Intervalle $x_0 \ldots x_1$ endlich bleibt, so ist für ein ausserhalb des Intervalles $-1 \ldots +1$ liegendes Werthepaar x_0, x_1

$$\int_{x_0}^{x_1} \left\{(1-x^2)\left(\frac{d\omega(x)}{dx}\right)^2 - n(n+2\nu)\,\omega^2(x)\right\}(1-x^2)^{\frac{2\nu-1}{2}}\,dx < 0$$

wenn $\omega(x)$ der Bedingung

$$\left[\frac{2\nu(1-x^2)^{\frac{2\nu+1}{2}}\,C_{n-1}^{\nu+1}(x)\,\omega^2(x)}{C_n^\nu(x)}\right]_{x_0}^{x_1} \leqq 0$$

genügt, erfüllt aber $\omega(x)$ die Relation

$$\left[\frac{2\nu(1-x^2)^{\frac{2\nu+1}{2}}\,C_{n-1}^{\nu+1}(x)\,\omega^2(x)}{C_n^\nu(x)}\right]_{x_0}^{x_1} \geqq 0$$

so besteht für jedes dem Intervalle $-1 \ldots +1$ angehörigen Werthepaare x_0, x_1, innerhalb dessen sich keine Wurzel der Gleichung $C_n^\nu(x) = 0$, befindet, die Gleichung

$$\int_{x_0}^{x_1} \left\{(1-x^2)\left(\frac{d\omega(x)}{dx}\right)^2 - n(n+2\nu)\,\omega^2(x)\right\}(1-x^2)^{\frac{2\nu-1}{2}}\,dx > 0.$$

4) Ist $\tilde{\omega}(x)$ die Anzahl der verschiedenen Primtheiler von x,

$(x) = (-1)^{\tilde{\omega}(x)}$, wenn x eine durch kein Quadrat (ausser 1) theilbare ganze Zahl ist, und gleich 0 in allen anderen Fällen,

$\varphi_k(x)$ die Anzahl der Systeme von k ganzen Zahlen des Intervalles $1 \ldots x$, welche ein zu x theilerfremdes Zahlensystem bilden,

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wenn $(-1)^m \left[\frac{e^{-x}\,x^{m+1}\;T_{n-1}^{m+1}(x)\,\omega^2(x)}{T_n^m(x)}\right]_{x_0}^{x_1} \geqq 0$ ist, und die ganze Function $T_n^m(x)$ durch die Gleichung

$$T_n^m(x) = \sum_{\lambda=0}^{\lambda=n} (-1)^\lambda \frac{x^{n-\lambda}}{\Pi(\lambda)\,\Pi(n-\lambda)\,\Pi(m+n-\lambda)}$$

definirt wird, sind aber x_0 und x_1 positiv und liegt zwischen ihnen keine Wurzel der Gleichung $T_n^m(x) = 0$, so ist

$$\int_{x_0}^{x_1} \left\{x\left(\frac{d\omega(x)}{dx}\right)^2 - n\omega^2(x)\right\} e^{-x}\,x^m\,dx > 0 \qquad (m > -1;\ n \text{ ganzzahlig, nicht negativ})$$

falls $\left[\frac{e^{-x}\,x^{m+1}\;T_{n-1}^{m+1}\,\omega^2(x)}{T_n^m(x)}\right]_{x_0}^{x_1} \geqq 0$ ist.

$\varphi(x, n)$ die Anzahl aller ganzen positiven n nicht überschreitenden Zahlen, welche zu x theilerfremd sind,

$\omega(x)$ die Anzahl der Zerlegungen der ganzen Zahl x in ein Product von zwei theilerfremden Factoren,

$f_\beta(x)$ die Anzahl der positiven ganzzahligen Lösungen der Gleichungen $n_1 n_2 \ldots n_\beta = x$,

$\chi(x)$ die Anzahl der Darstellungen von x als Summe von zwei Quadraten,

$\mu_r(x) = 0$, wenn x durch eine r^{te} Potenz (ausser 1) theilbar ist, und gleich 1 in allen anderen Fällen,

$\psi_k(x)$ die Summe der k^{ten} Potenzen der Theiler der ganzen Zahl x,

$\alpha(1) = 0$, $\alpha(x) = 0$, wenn x eine Primzahl in einer höheren als der zweiten, oder mehr als eine Primzahl in einer höheren als der ersten Potenz enthält,

$\alpha(x) = (-1)^{\bar{\omega}(x)}$, wenn x einen Primfactor in der zweiten, die anderen aber nur in der ersten Potenz enthält,

$\alpha(x) = (-1)^{\bar{\omega}(x)+1}\,\bar{\omega}(x)$, wenn x durch kein Quadrat (ausser 1) theilbar ist,

$\lambda(x)$ gleich $+1$ oder -1, je nachdem x aus einer geraden oder ungeraden Anzahl von gleichen oder verschiedenen Primzahlen zusammengesetzt ist,

$\bar{\psi}_k(x)$ die Summe der k^{ten} Potenzen derjenigen Divisoren der ganzen Zahl x, welche durch kein Quadrat (ausser 1) theilbar sind,

$\chi_{4k,\,2k-1}(x)$ der Überschuss der Anzahl derjenigen ungeraden Divisoren von x, welche die Form $4ks + \lambda$ $(\lambda \leqq 2k-1)$ haben, über die Anzahl der übrigen ungeraden Theiler,

$\left(\frac{\Delta}{x}\right)$ das Legendre-Jacobi'sche Symbol,

$\chi(\Delta, x)$ die Anzahl der Lösungen der Congruenz $y^2 \equiv \Delta \pmod{4x}$,

$\nu(x)$ gleich 0, wenn x keine Primzahlpotenz ist und gleich $\log x$ in allen anderen Fällen

$\Theta(x)$ die Anzahl der x nicht übersehreitenden Primzahlen

$\chi_n(x)$ gleich der über alle Divisoren der ganzen Zahl n ausgedehnten Summe $\sum_d \mu\left(\frac{n}{d}\right) C_d^{\nu}(x)$

und setzt man

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1} |\chi_\lambda(y_\mu)\,.\;C_\lambda^{\nu}(y_\mu)|_{(\lambda,\,\mu=1,2,\ldots,n)} \prod_1^n{}_k (1-y_k^2)^{\frac{2\nu-1}{2}}\, dy_k = A_n$$

so bestehen die Gleichungen:

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1} \begin{vmatrix} 0, & \mu(1), & \mu(2), \ldots, \mu(n) \\ C_1^{\nu}(y_1), & \chi_1(y_1), & \chi_2(y_1), \ldots, \chi_n(y_1) \\ C_2^{\nu}(y_2), & \chi_1(y_2), & \chi_2(y_2), \ldots, \chi_n(y_2) \\ \cdots & \cdots & \cdots \\ C_n^{\nu}(y_n), & \chi_1(y_n), & \chi_2(y_n), \ldots, \chi_n(y_n) \end{vmatrix} \left| C_\lambda^{\nu}(y_\mu) \right|_{(\lambda,\,\mu=1,2,\ldots,n)} \prod_1^n{}_k (1-y_k^2)^{\frac{2\nu-1}{2}}\, dy_k = A_n$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1} \begin{vmatrix} 0, & \varphi_k(1), & \varphi_k(2), \ldots, \varphi_k(n) \\ C_1^{\nu}(y_1), & \chi_1(y_1), & \chi_2(y_1), \ldots, \chi_n(y_1) \\ C_2^{\nu}(y_2), & \chi_1(y_2), & \chi_2(y_2), \ldots, \chi_n(y_2) \\ \cdots & \cdots & \cdots \\ C_n^{\nu}(y_n), & \chi_1(y_n), & \chi_2(y_n), \ldots, \chi_n(y_n) \end{vmatrix} \left| C_\lambda^{\nu}(y_\mu) \right|_{(\lambda,\,\mu=1,2,\ldots,n)} \prod_1^n{}_\varkappa (1-y_\varkappa^2)^{\frac{2\nu-1}{2}}\, dy_\varkappa =$$

$$= A_n\left(\frac{n^{k+1}}{k+1} + \frac{1}{2}\,n^k + \binom{k}{2}\frac{B_1}{k-1}\,n^{k-1} - \binom{k}{4}\frac{B_3}{k-3}\,n^{k-3} + \binom{k}{6}\frac{B_5}{k-5}\,n^{k-5} - \ldots\right)$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0, & \varphi(1,n), & \varphi(2,n), & \ldots, & \varphi(n,n) \\ C_1^\nu(y_1), & \chi_1(y_1), & \chi_2(y_1), & \ldots, & \chi_n(y_1) \\ C_2^\nu(y_2), & \chi_1(y_2), & \chi_2(y_2), & \ldots, & \chi_n(y_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ C_n^\nu(y_n), & \chi_1(y_n), & \chi_2(y_n), & \ldots, & \chi_n(y_n) \end{vmatrix} \, |C_\lambda^\nu(y_\mu)|_{(\lambda,\mu=1,2,\ldots,n)} \prod_1^n{}_k (1-y_k^2)^{\frac{2\nu-1}{2}} dy_k = \frac{6n^2A_n}{\pi^2}\left\{\log n + 2C-1\right\} + \varepsilon A_n\left(2n(\log n)^2 + n\log n\left(5C+\frac{\pi^2}{6}\right)+2\log n + 2C(2C-1)n + 2(2C-1) + \frac{\pi^2}{6}n^{\frac{3}{2}}\right) \quad {}^{1)}$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0, & \mu_2(1), & \mu_2(2), & \ldots, & \mu_2(n) \\ C_1^\nu(y_1), & \chi_1(y_1), & \chi_2(y_1), & \ldots, & \chi_n(y_1) \\ C_2^\nu(y_2), & \chi_1(y_2), & \chi_2(y_2), & \ldots, & \chi_n(y_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ C_n^\nu(y_n), & \chi_1(y_n), & \chi_2(y_n), & \ldots, & \chi_n(y_n) \end{vmatrix} \, |C_\lambda^\nu(y_\mu)|_{(\lambda,\mu=1,2,\ldots,n)} \prod_1^n{}_k (1-y_k^2)^{\frac{2\nu-1}{2}} dy_k = A_n \sum_{x=1}^{x=n} \omega(x)$$

$$= \frac{6A_n n}{\pi^2}\left\{\log n + \frac{12\mathfrak{F}}{\pi^2} + 2C-1\right\} + \varepsilon A_n\left\{\left(\frac{\log n}{2}+5+3C+2\log 2\right)\sqrt{n}+2\right\} \quad {}^{2)}$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0, & 1^k, & 2^k, & \ldots, & n^k \\ C_1^\nu(y_1), & \chi_1(y_1), & \chi_2(y_1), & \ldots, & \chi_n(y_1) \\ C_2^\nu(y_2), & \chi_1(y_2), & \chi_2(y_2), & \ldots, & \chi_n(y_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ C_n^\nu(y_n), & \chi_1(y_n), & \chi_2(y_n), & \ldots, & \chi_n(y_n) \end{vmatrix} \, |C_\lambda^\nu(y_\mu)|_{(\lambda,\mu=1,2,\ldots,n)} \prod_1^n{}_\varkappa (1-y_\varkappa^2)^{\frac{2\nu-1}{2}} dy_\varkappa = A_n \sum_{x=1}^{x=n} \psi_k(x)$$

$$= A_n n^{k+1}\zeta(k+1) + \varepsilon A_n n^k\left\{\frac{\zeta(k+1)}{(n+1)^{k-1}} + \varepsilon'\zeta(k)\right\} \quad {}^{3)}$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0, & 1, & 1, & \ldots, & 1 \\ C_1^\nu(y_1), & \chi_1(y_1), & \chi_2(y_1), & \ldots, & \chi_n(y_1) \\ C_2^\nu(y_2), & \chi_1(y_2), & \chi_2(y_2), & \ldots, & \chi_n(y_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ C_2^\nu(y_n), & \chi_1(y_n), & \chi_2(y_n), & \ldots, & \chi_n(y_n) \end{vmatrix} \, |C_\lambda^\nu(y_\mu)|_{(\lambda,\mu=1,2,\ldots,n)} \prod_1^n{}_k (1-y_k^2)^{\frac{2\nu-1}{2}} dy_k = A_n \sum_{x=1}^{x=n} \psi_0(x)$$

$$= A_n n(\log n + 2C-1) + 4\varepsilon\sqrt{n}A_n$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0, & \alpha(1), & \alpha(2), & \ldots, & \alpha(n) \\ C_1^\nu(y_1), & \chi_1(y_1), & \chi_2(y_2), & \ldots, & \chi_n(y_1) \\ C_2^\nu(y_2), & \chi_1(y_2), & \chi_2(y_2), & \ldots, & \chi_n(y_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ C_n^\nu(y_n), & \chi_1(y_n), & \chi_2(y_n), & \ldots, & \chi_n(y_n) \end{vmatrix} \, |C_\lambda^\nu(y_\mu)|_{(\lambda,\mu=1,2,\ldots,n)} \prod_1^n{}_k (1-y_k^2)^{\frac{2\nu-1}{2}} dy_k = A_n\Theta(n)$$

$$= \frac{A_n n}{\log n - 1} + \varepsilon A_n\sqrt{n}$$

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1) $C = 0\cdot57721566\ldots$ ist die bekannte Euler'sche Constante des Integrallogarithmus; $|\varepsilon|<1$.

2) $\mathfrak{F} = \sum_{y=2}^{x=\infty}\frac{\log x}{x^2} = 0\cdot9375482543\ldots$

3) $\zeta(s) = \sum_{n=1}^{n=\infty}\frac{1}{n^s}$; $|\varepsilon'|<1$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0, & f_\beta(1), & f_\beta(2), & \ldots, & f_\beta(n)\\ C_1^\nu(y_1), & \chi_1(y_1), & \chi_2(y_1), & \ldots, & \chi_n(y_1)\\ C_2^\nu(y_2), & \chi_1(y_2), & \chi_2(y_2), & \ldots, & \chi_n(y_2)\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ C_n^\nu(y_n), & \chi_1(y_n), & \chi_2(y_n), & \ldots, & \chi_n(y_n)\end{vmatrix} |C_\lambda^\nu(y_\mu)|_{(\lambda,\,\mu=1,2,\ldots,n)}\prod_{1}^{n}{}_{k}\,(1-y_k^2)^{\frac{2\nu-1}{2}}\,dy_k = A_n\sum_{x=1}^{x=n} f_{\beta+1}(x)$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0, & \lambda(1)\omega(1), & \lambda(2)\omega(2), & \ldots, & \lambda(n)\omega(n)\\ C_1^\nu(y_1), & \chi_1(y_1), & \chi_2(y_1), & \ldots, & \chi_n(y_1)\\ C_2^\nu(y_2), & \chi_1(y_2), & \chi_2(y_2), & \ldots, & \chi_n(y_2)\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ C_n^\nu(y_n), & \chi_1(y_n), & \chi_2(y_n), & \ldots, & \chi_n(y_n)\end{vmatrix} |C_\lambda^\nu(y_\mu)|_{(\lambda,\,\mu=1,2,\ldots,n)}\prod_{1}^{n}{}_{k}\,(1-y_k^2)^{\frac{2\nu-1}{2}}\,dy_k = $$

$$= A_n\sum_{x=1}^{x=n}\lambda(x).$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0, & \lambda(1)\chi(\Delta,1), & \lambda(2)\chi(\Delta,2), & \ldots, & \lambda(n)\chi(\Delta,n)\\ C_1^\nu(y_1), & \chi_1(y_1), & \chi_2(y_1), & \ldots & \chi_n(y_1)\\ C_2^\nu(y_2), & \chi_1(y_2), & \chi_2(y_2), & \ldots, & \chi_n(y_2)\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ C_n^\nu(y_n), & \chi_1(y_n), & \chi_2(y_n), & \ldots, & \chi_n(y_n)\end{vmatrix} |C_\lambda^\nu(y_\mu)|_{(\lambda,\,\mu=1,2,\ldots,n)}\prod_{1}^{n}{}_{k}\,(1-y_k^2)^{\frac{2\nu-1}{2}}\,dy_k =$$

$$2\,A_n\sum_{x=1}^{x=n}\lambda(x)\left(\frac{\Delta}{x}\right)$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0, & \mu(1), & 0, & 0, & \ldots, & \mu(2), & 0, & \ldots, & \mu(3), & \ldots, & \mu([\sqrt[r]{n}])\\ C_1^\nu(y_1), & \chi_1(y_1), & \chi_2(y_1), & \chi_3(y_1), & \ldots, & \chi_{2^r}(y_1), & \chi_{2^r+1}(y_1), & \ldots & \chi_{3^r}(y_1), & \ldots, & \chi_{[\sqrt[r]{n}]^r}(y_1)\\ C_2^\nu(y_2), & \chi_1(y_2), & \chi_2(y_2), & \chi_3(y_2), & \ldots & \chi_{2^r}(y_2), & \chi_{2^r+1}(y_2), & \ldots & \chi_{3^r}(y_2), & \ldots, & \chi_{[\sqrt[r]{n}]^r}(y_2)\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ C_n^\nu(y_n), & \chi_1(y_n), & \chi_2(y_n), & \chi_3(y_n), & \ldots & \chi_{2^r}(y_n), & \chi_{2^r+1}(y_n), & \ldots & \chi_{3^r}(y_n), & \ldots, & \chi_{[\sqrt[r]{n}]^r}(y_n)\end{vmatrix}.$$

$$.\,|C_\lambda^\nu(y_\mu)|_{(\lambda,\,\mu=1,2,\ldots,n)}\prod_{1}^{n}{}_{k}\,(1-y_k^2)^{\frac{2\nu-1}{2}}\,dy_k = A_n\sum_{x=1}^{x=n}\mu_r(x) = \frac{A_n n}{\zeta(r)} + \varepsilon A_n(1+\zeta(r))\,n^{\frac{1}{r}}$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0, & \nu(1), & \nu(2), & \ldots, & \nu(n)\\ C_1^\nu(y_1), & \chi_1(y_1), & \chi_2(y_1), & \ldots, & \chi_n(y_1)\\ C_2^\nu(y_2), & \chi_1(y_2), & \chi_2(y_2), & \ldots, & \chi_n(y_2)\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ C_n^\nu(y_n), & \chi_1(y_n), & \chi_2(y_n), & \ldots, & \chi_n(y_n)\end{vmatrix} |C_\lambda^\nu(y_\mu)|_{(\lambda,\,\mu=1,2,\ldots,n)}\prod_{1}^{n}{}_{k}\,(1-y_k^2)^{\frac{2\nu-1}{2}}\,dy_k = A_n\log\Pi(n)$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0, & 1, & 0, & -1, & 0, & \ldots, & \sin\frac{k\pi}{2}, & \ldots, & \sin\frac{n\pi}{2}\\ C_1^\nu(y_1), & \chi_1(y_1), & \chi_2(y_1), & \chi_3(y_1), & \chi_4(y_1), & \ldots, & \chi_k(y_1), & \ldots, & \chi_n(y_1)\\ C_2^\nu(y_2), & \chi_1(y_2), & \chi_2(y_2), & \chi_3(y_2), & \chi_4(y_2), & \ldots, & \chi_k(y_2), & \ldots & \chi_n(y_2)\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ C_n^\nu(y_n), & \chi_1(y_n), & \chi_2(y_n), & \chi_3(y_n), & \chi_4(y_n), & \ldots, & \chi_k(y_n), & \ldots, & \chi_n(y_n)\end{vmatrix}.$$

$$.\,|C_\lambda^\nu(y_\mu)|_{(\lambda,\,\mu=1,2,\ldots,n)}\prod_{1}^{n}{}_{k}\,(1-y_k^2)^{\frac{2\nu-1}{2}}\,dy_k = A_n\sum_{x=1}^{x=n}\chi(x) = \frac{A_n n\pi}{4} + 4\varepsilon A_n\sqrt{n}$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0\;, & \frac{\mu_2(1)}{1^{k_1}}, & \frac{\mu_2(2)}{2^{k_1}}, & \ldots, & \frac{\mu_2(n)}{n^{k_1}} \\ C_1^{\nu}(y_1), & \chi_1(y_1), & \chi_2(y_1), & \ldots, & \chi_n(y_1) \\ C_2^{\nu}(y_2), & \chi_1(y_2), & \chi_2(y_2), & \ldots & \chi_n(y_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ C_n^{\nu}(y_n), & \chi_1(y_n), & \chi_2(y_n), & \ldots, & \chi_n(y_n) \end{vmatrix} \left|C_{\lambda}^{\nu}(y_{\mu})\right|_{(\lambda,\,\mu=1,\,2,\,\ldots,\,n)} \prod_{1}^{n}{}_{k}\,(1-y_k^2)^{\frac{2\nu-1}{2}}\,dy_k = A \sum_{x=1}^{x=n} \psi_{-k_1}(x)$$

$$= \frac{2\Pi(2k_1+2)\,A_n\,n}{(2\pi)^{2k_1+2}\,B_{k_1+1}}\,\zeta(k_1+1) + \varepsilon A_n\left(\frac{\pi^4}{18} + \frac{1}{n^{k_1-1}}\right)\;(k_1>1)$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0\;, & \frac{\mu_2(1)}{1}, & \frac{\mu_2(2)}{2}, & \ldots, & \frac{\mu_2(n)}{n} \\ C_1^{\nu}(y_1), & \chi_1(y_1), & \chi_2(y_1), & \ldots, & \chi_n(y_1) \\ C_2^{\nu}(y_2), & \chi_1(y_2), & \chi_2(y_2), & \ldots, & \chi_n(y_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ C_n^{\nu}(y_n), & \chi_1(y_n), & \chi_2(y_n), & \ldots, & \chi_n(y_n) \end{vmatrix} \left|C_{\lambda}^{\nu}(y_{\mu})\right|_{(\lambda,\,\mu=1,\,2,\,\ldots,\,n)} \prod_{1}^{n}{}_{k}\,(1-y_k^2)^{\frac{2\nu-1}{2}}\,dy_k =$$

$$= \frac{15}{\pi^2}A_n n + \varepsilon A_n\left[\frac{\pi^2}{6}\left(\log n + C + \frac{\pi^2}{6}\right) + \frac{2}{\sqrt{n}}\right]$$

$$\int_{-1}^{+1}\int_{-1}^{+1}\cdots\int_{-1}^{+1}\begin{vmatrix} 0\;, & 1\;, & 0\;, & 1\;, & \ldots, & (-1)^{\left[\frac{\lambda-1}{k_1}\right]}, & 0\;, & (-1)^{\left[\frac{\lambda}{k_1}\right]} & 0\;, & (-1)^{\left[\frac{\lambda+1}{k_1}\right]}, & \ldots \\ C_1^{\nu}(y_1), & \chi_1(y_1), & \chi_2(y_1), & \chi_3(y_1), & \ldots, & \chi_{2\lambda-1}(y_1), & \chi_{2\lambda}(y_1), & \chi_{2\lambda+1}(y_1), & \chi_{2\lambda+2}(y_1), & \chi_{2\lambda+3}(y_1), & \ldots \\ C_2^{\nu}(y_2), & \chi_1(y_2), & \chi_2(y_2), & \chi_3(y_2), & \ldots, & \chi_{2\lambda-1}(y_2), & \chi_{2\lambda}(y_2), & \chi_{2\lambda-1}(y_2), & \chi_{2\lambda-2}(y_2), & \chi_{2\lambda+3}(y_2), & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_n^{\nu}(y_n), & \chi_1(y_n), & \chi_2(y_n), & \chi_3(y_n), & \ldots, & \chi_{2\lambda-1}(y_n), & \chi_{2\lambda}(y_n), & \chi_{2\lambda+1}(y_n), & \chi_{2\lambda+2}(y_n), & \chi_{2\lambda+3}(y_n), & \ldots \end{vmatrix}.$$

$$\left|C_{\lambda}^{\nu}(y_{\mu})\right|_{(\lambda,\,\mu=1,\,2,\,\ldots,\,n)} \prod_{1}^{n}{}_{k}\,(1-y_k^2)^{\frac{2\nu-1}{2}}\,dy_k = A_n \sum_{x=1}^{x=n} \chi_{4k_1,\,2k_1-1}(x) =$$

$$= \frac{A_n n\pi}{4k_1}\sum_{\mu=1}^{\mu=k_1}\operatorname{cotang}\frac{(2\mu-1)\pi}{4k_1} + \varepsilon A_n\left\{\frac{(2k_1+1)(2n+1)+4}{\sqrt{8n+1}-3} + \frac{3(2k_1+1)}{4}\right\}.$$