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# On Additive Functions Fulfilling some Additional Condition\*

Von

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#### Abstract

Let  $D \subset \mathbb{R}^2$  be an arbitrary set. We consider the following question: What kind of assumptions on D imply that every additive function  $f \quad \mathbb{R} \to \mathbb{R}$  satisfying the condition

$$(x, y) \in D \Rightarrow f(x)f(y) = 0 \tag{1}$$

is identically equal to zero? It is true if D is a non-empty open subset of  $\mathbb{R}^2$  G. Szabó posed this problem for  $D = \{(x, y); x^2 + y^2 = 1\}$  ([7]). We give an affirmative answer to Szabó's question and, moreover, we give some sufficient conditions to obtain the above assertion in much more general spaces.

1.

Let X and Y be linear spaces over the rationals  $\mathbb{Q}$ . A function  $f \quad X \to Y$  is called additive if it satisfies the (Cauchy's) equation

$$f(x+y) = f(x) + f(y), \quad x, y \in X.$$
 (2)

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Every additive function is uniquely determined by its values on a socalled Hamel base (i.e. a base of X over the rationals) and it fulfills the condition

 $f(rx) = rf(x), \quad x \in X, \quad r \in \mathbb{Q},$ 

(cf. [4], for example). We start our considerations with two examples.

**Example 1.** Let  $f \mathbb{R} \to \mathbb{R}$  ( $\mathbb{R}$  is here and in the sequel the set off all reals) be a discontinuous additive function vanishing on a saturated non-measurable in the Lebesgue sense subset S ([4] p. 58 and 297-Th. 7) and put  $D = (S \times \mathbb{R}) \cup (\mathbb{R} \times S)$ . Evidently f is not identically equal to zero and condition (1) is fulfilled.

The set D, though large in a sense, does not contain any segment. The following second example shows that even when D contains a segment it is possible to find a non-zero additive function fulfilling condition (1).

**Example 2.** Let  $f \ \mathbb{R}$  be a discontinuous additive function such that the restriction of f to the set of all rationals is equal to zero. If  $D = \{(x, y); \max\{|x|, |y|\} = 1\}$ , then f satisfies condition (1) and is not identically equal to zero.

The set from Example 2 is a unit circle if we treat  $\mathbb{R}^2$  as a linear space endowed with the norm  $||(x, y)|| := \max\{|x|, |y|\}$ . We shall show that the answer to our (Szabó's) question is positive if we take a different norm in  $\mathbb{R}^2$  We have the following

**Remark 1.** Let  $D = \{(x, y) \in \mathbb{R}^2; |x| + |y| = 1\}$  and let  $f \mathbb{R} \to \mathbb{R}$  be an additive function fulfilling condition (1). Then f is identically equal to zero.

Proof: According to our assumptions we obtain

$$f(x)f(1-x) = 0, \quad x \in (0,1).$$
 (3)

If f(1) = 0 it follows from (3) that  $f(x) = 0, x \in (0, 1)$ , and hence  $f \equiv 0$ . Assume

$$f(1) \neq 0 \tag{4}$$

and take an  $x_0 \in (0, 1)$  such that  $f(x_0) = 0$ . For arbitrary  $z \in \mathbb{R}$  there exists an integer *n* such that  $z + nx_0 \in (0, 1)$ . Thus

$$f(z + nx_0)f(1 - z - nx_0) = 0,$$

which, because of the additivity of f, yields the condition

$$f(z)f(1-z) = 0, \quad z \in \mathbb{R}.$$

Putting  $-\gamma$  instead of  $\gamma$  in this equality and adding both we get

$$f(\chi)f(1)=0,$$

which contradicts (4) and proves Remark 1.

A positive answer to Szabó's question is contained in

**Theorem 1.** Let  $f : \mathbb{R} \to \mathbb{R}$  be an additive function fulfilling condition (1) where  $D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$ . Then f is identically equal to zero.

*Proof*: Take an arbitrary  $x \in (0, 1)$  and choose a y such that  $x^2 + y^2 = 1$ . Setting

$$u = \frac{3x + 4y}{5}, \quad v = \frac{4x - 3y}{5}$$

we observe that

$$u^2 + v^2 = x^2 + y^2 = 1$$

By virtue of (1)

$$f(u)f(v) = f(x)f(y) = 0.$$
 (5)

Moreover, by (5)

$$0 = f(u)f(v) = \frac{1}{25}[3f(x) + 4f(y)][4f(x) - 3f(y)]$$
$$= \frac{12}{25}(f(x)^2 - f(y)^2)$$

and hence  $f(x)^2 = f(y)^2$  On account of (5) f(x) = 0. Due to the arbitraryness of  $x \in [0, 1)$ , f is identically equal to zero because it is additive.

**Corollary 1.** A similar result holds true if  $D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = r^2\}$ , where r > 0 is an arbitrary constant.

*Proof*: The function  $F(x) = f(rx), x \in \mathbb{R}$  fulfills all assumptions of Theorem 1. We have also

**Theorem 2.** Let X be a real normed space and let Y be an arbitrary linear space. If  $f X \to Y$  is an arbitrary additive function fulfilling the condition

$$||x||^{2} + ||y||^{2} = 1 \Rightarrow f(x) = 0 \text{ or } f(y) = 0,$$

then f is identically equal to zero.

*Proof*: First let us assume that dim X > 1. Take an arbitrary  $x \in X$  such that  $||x|| = \frac{\sqrt{2}}{2}$  and put y = x. Then

$$\|x\|^2 + \|y\|^2 = 1$$

and by our assumption we get

f(x) = 0,

which means that f vanishes on a circle  $C = \{x \in X | \|x\| = \frac{\sqrt{2}}{2}\}$ . Since dim  $X \ge 2$  for every  $u \in X$ ,  $\|u\| < \frac{\sqrt{2}}{2}$ , there exist  $v_1, v_2 \in C$  such that  $v_1 + v_2 = u$  ([1], see the proof of Lemma 1). Consequently

$$f(u) = f(v_1 + v_2) = f(v_1) + f(v_2) = 0.$$

Thus f, being an additive function vanishing on a ball, has to be identically equal to zero.

If dim X = 0, the assertion is trivial. If, finally, dim X = 1 we may assume that  $X = \mathbb{R}$  and that  $||x|| = r^{-1}|x|$  for some r > 0. Thus for every linear functional  $\varphi \quad Y \to \mathbb{R}$  the function  $\varphi \circ f \quad X \to \mathbb{R}$  satisfies the assumptions of Corollary 1, implying that  $\varphi \circ f = 0$ . But the linear functionals on Y separate the points of Y. Thus f = 0.

Let G be an abelian group and let K be a field of characteristic zero. For mappings  $w \quad G \to \mathbb{K}$  and an element  $h \in G$  the difference operator  $\Delta_h$  is defined by

$$\Delta_b w(x) := w(x+b) - w(x).$$

A mapping  $w \in G \to \mathbb{K}$  is called a generalized polynomial of degree less than n + 1 iff

$$\Delta_h^{n+1}w(x) = 0, \quad x, b \in G,$$

where  $\Delta^k$  denotes the k - th iterate of  $\Delta$ .

**Theorem 3.** Let  $f \quad G \to \mathbb{K}$  be an additive function and let

$$D = \{(v(x), w(x)) \in \mathbb{K} \times \mathbb{K}; x \in G)\},\$$

where  $v, w \in G \to \mathbb{K}$  are generalized polynomials such that  $\lim_{\mathbb{Q}} v(G) = \lim_{\mathbb{Q}} w(G) = \mathbb{K}$ . If f fulfills condition (1), then it is identically equal to zero.

Proof: By our assumptions

$$f(v(x))f(w(x)) = 0, \quad x \in G.$$
(6)

Since  $f \circ v$  and  $f \circ w$  are generalized polynomials we can apply a result of F. Halter-Koch, L. Reich and J. Schwaiger ([3], Th. 2). Therefore

 $f \circ v \equiv 0$  or  $f \circ w \equiv 0$ . It follows from the equality  $\lim_{\mathbb{Q}} v(G) = \lim_{\mathbb{Q}} w(G) = \mathbb{K}$  that f is identically equal to zero.

**Remark 2.** The assumption  $\lim_{\mathbb{Q}} v(G) = \lim_{\mathbb{Q}} w(G) = \mathbb{K}$  is essential in Theorem 3.

This can be seen by taking v = id and w = f, where f is a function as defined in Example 2.

**Corollary 2.** Let  $v, w \in \mathbb{R} \to \mathbb{R}$  be arbitrary (ordinary) polynomials of degree at least one. If  $f \in \mathbb{R} \to \mathbb{R}$  is an additive function fulfilling condition (6) then it is identically equal to zero.

This is so, since  $v(\mathbb{R})$  and  $w(\mathbb{R})$  are non-trivial intervals.

Condition (1) may be generalized by replacing the righthand side of the implication (i.e. f(x)f(y) = 0 for  $(x, y) \in D$ ) by Q(f(x), f(y)) = 0 for all  $(x, y) \in D$ , where Q is a polynomial in indeterminates X and Y over  $\mathbb{R}(Q \in \mathbb{R}[X, Y])$ . This means that we now are interested in conditions on  $D \subseteq \mathbb{R}^2$  such that

$$(x, y) \in D \Rightarrow \mathcal{Q}(f(x), f(y)) = 0 \tag{1'}$$

for an additive function  $f \mathbb{R} \to \mathbb{R}$  implies f = 0.

In this situation we will show

**Theorem 3'.** Let  $f \ \mathbb{R} \to \mathbb{R}$  be additive and let p and q be generalized polynomials of degree 1, i.e. p = g + a, q = b + b, where  $g, b \ \mathbb{R} \to \mathbb{R}$ are additive and a, b real constants. Assume that  $p(\mathbb{R})$  and  $q(\mathbb{R})$  contain Hamel bases. Furthermore, let  $Q \in \mathbb{R}[X, Y]$  such that no polynomial AX + BY + C with  $AB \neq 0$  divides Q(X, Y), and let

$$D:=\{(p(u),q(u))|u\in\mathbb{R}\}\subseteq\mathbb{R}^2$$

Then, if

$$(x, y) \in D \Rightarrow \mathcal{Q}(f(x), f(y)) = 0, \tag{1'}$$

we have f = 0.

Proof: We have  $f(p(u)) = (f \circ g)(u) + c$ ,  $f(q(u)) = (f \circ b)(u) + d$ , where  $c = f(a), d = f(b).f \circ g$  and  $f \circ b$  are additive, and since  $p(\mathbb{R}), q(\mathbb{R})$  contain Hamel bases, the same holds for  $g(\mathbb{R}), b(\mathbb{R})$ , and so  $f \circ g \neq 0, f \circ b \neq 0$ . By (1') we have

$$\mathcal{Q}((f \circ g)(u) + c, (f \circ b)(u) + d) = 0, \quad u \in \mathbb{R}.$$

We denote by  $Q_1(X, Y)$  the polynomial  $Q_1(X, Y) := Q(X + c, Y + d)$ , where  $Q_1 \neq 0, Q_1((f \circ g)(u), (f \circ b)(u)) = 0, u \in \mathbb{R}$ .

By [6, theorem 1] we get that  $f \circ g$ ,  $f \circ b$  are linearly dependent over  $\mathbb{R}$ , i.e. there exists  $(\lambda, \mu) \in \mathbb{R}^2$ ,  $(\lambda, \mu) \neq (0, 0)$  such that

$$\lambda(f \circ g) + \mu(f \circ b) = 0.$$
(7)

Since  $f \circ g \neq 0$ ,  $f \circ b \neq 0$  we deduce that  $\lambda \neq 0$ ,  $\mu \neq 0$ . But then by [6, theorem 2] we see that

$$\lambda X + \mu Y | \mathcal{Q}_1(X, Y),$$

and therefore

$$\lambda X + \mu Y - (\lambda \iota + \mu d) | \mathcal{Q}(X, Y),$$

where  $\lambda \mu \neq 0$ , which contradicts the assumption of the theorem. So we have necessarily f = 0, which concludes the proof.

The set D from Example 1 is large in a certain sense; it is saturated nonmeasurable in the Lebesgue sense as well as it is a second category set without Baire property. However, we prove the following

**Theorem 4.** Let  $f \ \mathbb{R}^n \to \mathbb{R}$  be an additive function fulfilling condition (1) and assume that  $D \subset \mathbb{R}^{2n}$  is a Lebesgue measurable subset with positive measure. Then f is identically equal to zero.

Proof: The set

$$H := \{x \in \mathbb{R}^n; f(x) = 0\}$$

is a subgroup of  $\mathbb{R}^n$  and since  $D \subset (H \times \mathbb{R}^n) \cup (\mathbb{R}^n \times H)$  the outer Lebesgue measure of H is positive. It is not hard to check hat H is dense in  $\mathbb{R}^n$  By Smítal's lemma ([4], [5]) the set  $G := (H \times H) + D$  is of full Lebesgue measure in  $\mathbb{R}^{2n}$  (in fact; since  $\mathbb{R}^n$  is separable there exists a countable subset  $H_0$  of H which is dense in  $\mathbb{R}^n$ , and by Smítal's lemma the set  $(H_0 \times H_0) + D$  has full Lebesgue measure in  $\mathbb{R}^{2n}$  and, of course,  $(H_0 \times H_0) + D \subset G$ ). Moreover, for every  $(x, y) \in G$  we have  $x = b_1 + d_1$ ,  $y = b_2 + d_2$ ,  $b_1$ ,  $b_2 \in H$ ,  $(d_1, d_2) \in D$ , and hence  $f(x)f(y) = f(d_1)$  $f(d_2) = 0$ . Therefore

$$G \subset (H \times \mathbb{R}^n) \cup (\mathbb{R}^n \times H) =: S.$$

We will show that H is measurable in the Lebesgue sense and of the full measure in  $\mathbb{R}^{2n}$  By Fubini's theorem the set

$$B := \{x \in \mathbb{R}^n; S_x = \{y \in \mathbb{R}^n; (x, y) \in S\} \text{ is measurable} \}$$

is measurable in the Lebesgue sense and of full measure in  $\mathbb{R}^n$ . If  $B \subset H$ , then H is measurable and of the full measure in  $\mathbb{R}^n$ . If  $B \setminus H \neq \emptyset$ , take an  $x \in B \setminus H$ . Then  $S_x = H$  and  $x \in B$ . So, H is measurable, too. Thus H,

being a dense subgroup of full measure in  $\mathbb{R}^n$ , is equal to  $\mathbb{R}^n$ . (In fact, any subgroup of  $\mathbb{R}^n$  of positive measure equals  $\mathbb{R}^n$ : Assume that H is a full measure group in  $\mathbb{R}^n$  Take an arbitrary x from  $\mathbb{R}^n$  Then the set x - H is also full measure in  $\mathbb{R}^n$  and therefore by the Steinhaus theorem the intersection  $H \cap (x - H)$  is a nonempty set. Choosing a z from this intersection we get that x = z + (x - z) belongs to H + H = H. Thus  $H = \mathbb{R}^n$ .)

The proof of Theorem 4 is finished.

A topological analogue of Theorem 4 is also true. One can prove the following

**Theorem 5.** Let *D* be a second category subset of  $\mathbb{R}^{2n}$  with the Baire property and let  $f \mathbb{R}^n \to \mathbb{R}$  be an additive function fulfilling condition (1). Then *f* is identically equal to zero.

*Proof*: The proof is quite similar to the proof of Theorem 4 because Fubini's theorem and Smítal's lemma have topological analogues ([2], [4]).

The results of Remark 1 and Theorem 1 can be viewed as special cases of the following.

**Theorem 6.** Let  $u, v \quad T \to \mathbb{R}$  be such that for all  $t \in T$  there is some  $t_1 \in T$  and some  $2 \times 2$ -matrix Q with rational and nonvanishing entries a, b, c, d such that  $(u(t_1), v(t_1))^T = Q(u(t), v(t))^T$  Moreover let u(T) or v(T) generate  $\mathbb{R}$  as a  $\mathbb{Q}$ -vector space. Then we have that the condition

$$(f \circ u) \ (f \circ v) = 0$$

implies f = 0.

*Proof*: Fix  $t \in T$  Without loss of generality we may suppose that f(u(t)) = 0. Choosing  $t_1$  and Q as above and using the fact that  $f(u(t_1)) = f(v(t_1)) = 0$  we get

$$0 = f(au(t) + bv(t))f(cu(t) + dv(t))$$
  
=  $acf(u(t))^{2} + adf(u(t))f(v(t)) + bcf(v(t))f(u(t)) + bdf(v(t))^{2}$   
=  $bdf(v(t))^{2}$ ,

implying that f(v(t)) = 0. Thus  $f \circ u = f \circ v = 0$  which gives us the desired result.

**Remark 3.** Using  $u = \cos$  and  $v = \sin$  we get Theorem 1 with  $t_1 = t + t_0$  where  $t_0$  is such that  $\cos(t_0) = 3/5$  and  $\sin(t_0) = 4/5$ , for example. Remark 1 may be considered as the case T = ]0, 1[, u(t) = t, v(t) = 1 - t, a = b = c = d = 1/2.

A different example (hyperbola) is given by  $u = \cosh$ ,  $v = \sinh$ ,  $t_1 = t + t_0$ , where now  $t_0$  is choosen in such a way that both  $\cosh(t_0)$  and  $\sinh(t_0)$  are positive rationals (which of course is possible).

#### References

- [1] Ger, R.: Some remarks on quadratic functionals. Glasnik Mat. 23, 315-330 (1988).
- [2] Ger, R., Kominek, Z. K., Sablik, M.: Generalized Smital's Lemma and a Theorem of Steinhaus. Radovi Mat. 1, 101–119 (1985).
- [3] Halter-Koch, F., Reich, L., Schwaiger, J.: On products of additive functions. Aequationes Math. 45, 83–88 (1993).
- [4] Kuczma, M.: An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality. Polish Scientific Publishers and Silesian University Press, Warszawa-Kraków-Katowice, 1985.
- [5] Kuczma, M., Smítal, J.: On measures connected with the Cauchy equation. Aequationes Math. 14, 421–428 (1976).
- [6] Reich, L., Schwaiger, J.: On polynomials in additive and multiplicative functions. In: J. Aczél (ed.) Functional Equations: History, Applications and Theory. D. Reidel Publishing Company, Dordrecht/Boston/Lancaster, 1984, 127–160.
- [7] Szabó, G.: Report of Meeting TheThirtieth International Symposium on Functional Equations, Oberwolfach, Germany 1992. Aequationes Math. 46, p. 294 (1993).

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