# On Gauss-Pólya’s Inequality 

By

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#### Abstract

Let $g, b \quad[a, b] \rightarrow \mathbf{R}$ be nonnegative nondecreasing functions such that $g$ and $b$ have a continuous first derivative and $g(a)=b(a), g(b)=b(b)$. Let $p=\left(p_{1}, p_{2}\right)$ be a pair of positive real numbers $p_{1}, p_{2}$ such that $p_{1}+p_{2}=1$. a) If $f[a, b] \rightarrow \mathbf{R}$ be a nonnegative nondecreasing function, then for $r, s<1$ $M_{p}^{[r]}\left(\int_{a}^{b} g^{\prime}(t) f(t) d t, \int_{a}^{b} b^{\prime}(t) f(t) d t\right) \leq \int_{a}^{b}\left(M_{p}^{[s]}(g(t), b(t))\right)^{\prime} f(t) d t$


holds, and for $r, s>1$ the inequality is reversed.
b) If $f[a, b] \rightarrow \mathbf{R}$ is a nonnegative nonincreasing function then for $r<1<s(1)$ holds and for $r>1>s$ the inequality is reversed.
Similar results are derived for quasiarithmetic and logarithmic means.
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## 1. Introduction

Gauss mentioned the following result in [2]:
If $f$ is a nonnegative and decreasing function then

$$
\begin{equation*}
\left(\int_{0}^{\infty} x^{2} f(x) d x\right)^{2} \leq \frac{5}{9} \int_{0}^{\infty} f(x) d x \int_{0}^{\infty} x^{4} f(x) d x \tag{2}
\end{equation*}
$$

Pólya and Szegö classical book "Problems and Theorems in Analysis, I" [7] gives the following generalization and extension of Gauss' inequality (2).

Theorem A. (Pólya's inequality) Let a and b be nonnegative real numbers.
a) If $f:[0, \infty) \rightarrow \mathbf{R}$ is a nonnegative and decreasing function, then

$$
\begin{align*}
\left(\int_{0}^{\infty} x^{a+b} f(x) d x\right)^{2} \leq & \left(1-\left(\frac{a-b}{a+b+1}\right)^{2}\right) \int_{0}^{\infty} x^{2 a} f(x) d x \\
& \times \int_{0}^{\infty} x^{2 b} f(x) d x \tag{3}
\end{align*}
$$

whenever the integrals exist.
b) If $f:[0,1) \rightarrow \mathbf{R}$ is a nonnegative and increasing function, then

$$
\begin{align*}
\left(\int_{0}^{1} x^{a+b} f(x) d x\right)^{2} \geq & \left(1-\left(\frac{a-b}{a+b+1}\right)^{2}\right) \int_{0}^{1} x^{2 a} f(x) d x \\
& \times \int_{0}^{1} x^{2 b} f(x) d x \tag{4}
\end{align*}
$$

Obviously, putting $a=0$ and $b=2$ in (3) we obtain Gauss' inequality. Recently Pec̆arić and Varošanec [6] obtained a generalization.

Theorem B. Let $f[a, b] \rightarrow \mathbf{R}$ be nonnegative and increasing, and let $x_{i} \quad[a, b] \rightarrow \mathbf{R}(i=1, \ldots, n)$ be nonnegative increasing functions with a continuous first derivative. If $p_{i},(i=1, \ldots, n)$ are positive real numbers such that $\sum_{i=1}^{n} \frac{1}{p_{i}}=1$, then

$$
\begin{equation*}
\int_{a}^{b}\left(\prod_{i=1}^{n}\left(x_{i}(t)\right)^{1 / p_{i}}\right)^{\prime} f(t) d t \geq \prod_{i=1}^{n}\left(\int_{a}^{b} x_{i}^{\prime}(t) f(t) d t\right)^{1 / p_{i}} \tag{5}
\end{equation*}
$$

If $x_{i}(a)=0$ for all $i=1, \ldots, n$ and of $f$ is a decreasing function then the reverse inequality bolds.

The previous result is an extension of the Pólya's inequality. If we substitute in (5): $n=2, p_{1}=p_{2}=2, a=0, b=1, g(x)=x^{2 n+1}, b(x)=$ $x^{2 \nu+1}$ where $u, v>0$, we have (4).

In this paper we provide generalizations of Theorem B in a number of directions. In Section 2 we first provide the inequality for weighted means. We note that, as is suggested by notation for means, our result extends to the case when the ordered pair $\left(p_{1}, p_{2}\right)$ is replaced by an $n$-tuple. We derive also a version of our theorem for higher derivatives.

Section 4 treats some corresponding results when $M$ is replaced by quasiarithmetic mean. This can be done when the function involved enjoys appropriate convexity properties. A second theorem in Section 4 allows one weight $p_{1}$ to be positive and the others negative.

Section 5 addresses the logarithmic mean.

## 2. Results Connected with Weighted Means

$M_{p}^{[s]}(a)$ denotes the weighted mean of order $r$ and weights $p=$ $\left(p_{1}, \ldots, p_{n}\right)$ of a positive sequence $a=\left(a_{1}, \ldots, a_{n}\right)$. The $n$-tuple $p$ is of positive numbers $p_{i}$ with $\sum_{1=i}^{n} p_{i}=1$. The mean is defined by

$$
M_{p}^{[r]}(a)=\left\{\begin{array}{cl}
\left(\sum_{i=1}^{n} p_{i} a_{i}^{r}\right)^{1 / r} & \text { for } r \neq 0 \\
\prod_{i=1}^{n} a_{i}^{p_{i}} & \text { for } r=0
\end{array}\right.
$$

In the special cases $r=-1,0,1$ we obtain respectively the familiar harmonic, geometric and arithmetic mean.

The following theorem, which is a simple consequence of Jensen's inequality for convex functions, is one of the most important inequalities between means.

Theorem C. If a andp are positive $n$-tuples and $s<t, s, t \in \mathbf{R}$, then

$$
\begin{equation*}
M_{p}^{[s]}(a) \leq M_{p}^{[t]}(a) \quad \text { for } \quad s<t \tag{6}
\end{equation*}
$$

with equality if and only if $a_{1}==a_{n}$.
A well-known consequence of the above statement is the inequality between arithmetic and geometric means. Previous results and refinements can be found in [3].

The following theorem is the generalization of Theorem B.
Theorem 1. Let $g, b \quad[a, b] \rightarrow \mathbf{R}$ be nonnegative nondecreasing functions such that $g$ and $b$ bave a continuous first derivative and $g(a)=b(a), g(b)=b(b)$. Let $p=\left(p_{1}, p_{2}\right)$ be a pair of positive real numbers $p_{1}, p_{2}$ such that $p_{1}+p_{2}=1$.
a) Iff $[a, b] \rightarrow \mathbf{R}$ be a nonnegative nondecreasing function, then for $r, s<1$

$$
\begin{equation*}
M_{p}^{[r]}\left(\int_{a}^{b} g^{\prime}(t) f(t) d t, \int_{a}^{b} b^{\prime}(t) f(t) d t\right) \leq \int_{a}^{b}\left(M_{p}^{[s]}(g(t), b(t))\right)^{\prime} f(t) d t \tag{7}
\end{equation*}
$$

bolds, and for $r, s>1$ the inequality is reversed.
b) Iff $[a, b] \rightarrow \mathbf{R}$ is a nonnegative nonincreasing function then for $r<1<s$ (7) bolds and for $r>1>s$ the inequality is reversed.

Proof: Let us suppose that $r, s<1$ and $f$ is nondecreasing. Using inequality (6) we obtain

$$
\begin{aligned}
& M_{p}^{[r]}\left(\int_{a}^{b} g^{\prime}(t) f(t) d t, \int_{a}^{b} b^{\prime}(t) f(t) d t\right) \\
& \leq M_{p}^{[1]}\left(\int_{a}^{b} g^{\prime}(t) f(t) d t, \int_{a}^{b} b^{\prime}(t) f(t) d t\right) \\
& =\int_{a}^{b}\left(p_{1} g^{\prime}(t)+p_{2} h^{\prime}(t)\right) f(t) d t \\
& =f(b) M_{p}^{[1]}(g(b), b(b))-f(a) M_{p}^{[1]}(g(a), b(a)) \\
& -\int_{a}^{b} M_{p}^{[1]}(g(t), h(t)) d f(t) \\
& \leq f(b) M_{p}^{[1]}(g(b), h(b))-f(a) M_{p}^{[1]}(g(a), h(a)) \\
& -\int_{a}^{b} M_{p}^{[s]}(g(t), h(t)) d f(t) \\
& =f(b) M_{p}^{[1]}(g(b), b(b))-f(a) M_{p}^{[1]}(g(a), b(a)) \\
& -\left(f(b) M_{p}^{[s]}(g(b), h(b))-f(a) M_{p}^{[s]}(g(a), b(a))\right. \\
& \left.-\int_{a}^{b}\left(M_{p}^{[f]}(g(t), b(t))\right)^{\prime} f(t) d t\right) \\
& =f(b)\left(M_{p}^{[1]}(g(b), h(b))-M_{p}^{[s]}(g(b), b(b))\right) \\
& -f(a)\left(M_{p}^{[1]}(g(a), b(a))-M_{p}^{[s]}(g(a), b(a))\right) \\
& +\int_{a}^{p}\left(M_{p}^{[s]}(g(t), b(t))\right)^{\prime} f(t) d t \\
& =\int_{a}^{b}\left(M_{p}^{[s]}(g(t), h(t))\right)^{\prime} f(t) d t .
\end{aligned}
$$

A similar proof applies in each of the other cases. $\quad \square$
Remark 1. In Theorem 1 we deal with two functions $g$ and $b$. Obviously a similar result holds for $n$ functions $x_{1}, ., x_{n}$ which satisfy the same conditions as $g$ and $b$.

Remark 2. It is obvious that on substituting $r=s=0$ into (7) we have inequality (5) for $n=2$. The result for $r=s=0$ is given in [1].

In the following theorem we consider an inequality involving higher derivatives.

Theorem 2. Let $f \quad[a, b] \rightarrow \mathbf{R}, x_{i} \quad[a, b] \rightarrow \underset{R}{\mathbf{R}}(i=1, \ldots, m)$ be nonnegative functions with continuous $n$-th derivatives such that $x_{i}^{(n)},(i=1, \ldots, m)$ are nonnegative functions and $p_{i},(i=1, \ldots, m)$ be positive real numbers such that $\sum_{i=1}^{\prime \prime \prime} p_{i}=1$.
a) If $(-1)^{n-1} f^{(n)}$ is a nonnegative function, then for $r, s<1$

$$
\begin{align*}
& M_{p}^{[r]}\left(\int_{a}^{b} x_{1}^{(n)}(t) f(t) d t, \ldots, \int_{a}^{b} x_{m}^{(n)}(t) f(t) d t\right)  \tag{8}\\
& \leq \Delta+\int_{a}^{b}\left(M_{p}^{[s]}\left(x_{1}(t), \ldots, x_{m}(t)\right)\right)^{(n)} f(t) d t
\end{align*}
$$

bolds, where

$$
\begin{aligned}
\Delta= & \sum_{k=0}^{n-1}(-1)^{n-k-1} f^{(n-k-1)}(t) \\
& \left.\left(\sum_{i=1}^{m} p_{i} x_{i}^{(k)}(t)-\left(M_{p}^{[s]}\left(x_{1}(t), \ldots, x_{m}(t)\right)\right)^{(k)}\right)\right|_{a} ^{b}
\end{aligned}
$$

If

$$
\begin{equation*}
x_{i}^{(k)}(a)=x_{j}^{(k)}(a) \text { and } x_{i}^{(k)}(b)=x_{j}^{(k)}(b) \text { for } i, j \in\{1, \ldots, m\} \tag{9}
\end{equation*}
$$

and $k=0, \ldots, n-1$, then

$$
\begin{align*}
& M_{p}^{[r]}\left(\int_{a}^{b} x_{1}^{(n)}(t) f(t) d t, \ldots, \int_{a}^{b} x_{m}^{(n)}(t) f(t) d t\right)  \tag{10}\\
& \quad \leq \int_{a}^{b}\left(M_{p}^{[s]}\left(x_{1}(t), \ldots x_{m}(t)\right)\right)^{(n)} f(t) d t .
\end{align*}
$$

If $r, s>1$, then the inequalities (8) and (10) are reversed.
b) If $(-1)^{n} f^{(n)}$ is a nonnegativefunction, then for $r<1<s$ the inequalities (8) and (10) bold and for $r>1>s$ they are reversed.

Proof: a) Let $r$ and $s$ be less than 1. Integrating by part $n$-times and using (6), we obtain

$$
\begin{aligned}
& M_{p}^{[r]}\left(\int_{a}^{b} x_{1}^{(n)}(t) f(t) d t, \quad \int_{a}^{b} x_{m}^{(n)}(t) f(t) d t\right) \\
& \leq M_{p}^{[1]}\left(\int_{a}^{b} x_{1}^{(n)}(t) f(t) d t, \ldots, \int_{a}^{b} x_{m}^{(n)}(t) f(t) d t\right) \\
&=\left.\left(\sum_{k=0}^{n-1}(-1)^{n-k 1} f^{(n-k-1)}(t) \sum_{i=1}^{m} p_{i} x_{i}^{(k)}(t)\right)\right|_{a} ^{b} \\
&-\int_{a}^{b} M_{p}^{[1]}\left(x_{1}(t), \ldots, x_{m}(t)\right)(-1)^{(n-1)} f^{(n)}(t) d t \\
& \leq\left.\left(\sum_{k=0}^{n-1}(-1)^{n-k 1} f^{(n-k-1)}(t) \sum_{i=1}^{m} p_{i} x_{i}^{(k)}(t)\right)\right|_{a} ^{b} \\
&-\int_{a}^{b} M_{p}^{[s]}\left(x_{1}(t), ., x_{m}(t)\right)(-1)^{(n-1)} f^{(n)}(t) d t \\
&= \Delta+\int_{a}^{b}\left(M_{p}^{[5]}\left(x_{1}(t), ., x_{m \prime}(t)\right)\right)^{(n)} f(t) d t .
\end{aligned}
$$

We shall prove that $\Delta=0$ if $x_{i}, i=1, . \quad, m$, satisfy (9).
Let us use notation $A_{k}=x_{i}^{(k)}(a)$ for $k=0,1, \ldots, n-1$. Then $\sum_{i=1}^{m \prime} p_{i} x_{i}^{(k)}(a)=A_{k}$. Consider the $k$-th order derivative of function $y^{p}$ where $y$ is an arbitrary function with $k$-th order derivative. First, there exists function $\phi_{k}^{[p]}$ such that

$$
\left(y^{p}\right)^{(k)}=\phi_{k}^{[p]}\left(y, y^{\prime}, . ., y^{(k)}\right)
$$

This follows by induction on $k$. For $k=1$ we have $\left(y^{p}\right)^{\prime}=p y^{p-1} y^{\prime}=$ $\phi_{1}^{[p]}\left(y, y^{\prime}\right)$. Suppose that proposition is valid for all $j<k+1$. Then using Leibniz's rule we get

$$
\begin{align*}
\left(y^{p}\right)^{(k+1)} & =\left(p y^{p-1} y^{\prime}\right)^{(k)} \\
& =p \sum_{j=0}^{k}\binom{k}{j}\left(y^{p-1}\right)^{(j)}\left(y^{\prime}\right)^{(k-j)} \\
& =p \sum_{j=0}^{k}\binom{k}{j} \phi_{j}^{[p-1]}\left(y, y^{\prime}, . \quad, y^{(j)}\right) y^{(k-j+1)}  \tag{11}\\
& =\phi_{k+1}^{[p]}\left(y, y^{\prime}, \ldots, y^{(k+1)}\right) .
\end{align*}
$$

Suppose that $s \neq 0$ and use the abbreviated notation $M(t)$ for the mean $M_{p}^{[s]}\left(x_{1}(t), \ldots, x_{m}(t)\right)$. Then $M^{s}(t)=\sum_{i=1}^{\prime \prime \prime} P_{i} x_{i}^{s}(t)$. The statement " $M^{(k)}(a)=A_{k}$ " will be proved by induction on $k$. It is easy to check for $k=0$ and $k=1$.

Suppose it holds for all $j<k+1$. Then

$$
\begin{aligned}
\left.\left(\sum_{i=1}^{m} p_{i} x_{i}^{s}(t)\right)^{(k+1)}\right|_{t=a}= & \left.\sum_{i=1}^{m} p_{i} \phi_{(k+1)}^{[s]}\left(x_{i}(t), x_{i}^{\prime}(t), \ldots, x_{i}^{(k+1)}(t)\right)\right|_{t=a} \\
= & \phi_{(k+1)}^{[s]}\left(A_{0}, A_{1}, ., A_{k+1}\right) \\
= & s \sum_{j=0}^{k}\binom{k}{j} \phi_{j}^{[s-1]}\left(A_{0}, A_{1}, \ldots, A_{j}\right) A_{k-j+1} \\
& +\phi_{k}^{[s-1]}\left(A_{0}, A_{1}, \quad, A_{k}\right) A_{k+1}
\end{aligned}
$$

On the other hand, using (11) we get

$$
\begin{aligned}
& \left.\left(M^{s}(t)\right)^{(k+1)}\right|_{t=a}=s \sum_{j=0}^{k}\binom{k}{j} \phi_{j}^{[s-1]}\left(M(a), M^{\prime}(a), \quad ., M^{(j)}(a)\right) \\
& \quad \times M^{(k-j+1)}(a)+\phi_{k}^{[s-1]}\left(M(a), M^{\prime}(a), \ldots, M^{(k)}(a)\right) M^{(k+1)}(a) \\
& =s \sum_{j=0}^{k}\binom{k}{j} \phi_{j}^{[s-1]}\left(A_{0}, A_{1}, \ldots, A_{j}\right) A_{k-j+1}+\phi_{k}^{[s-1]} \\
& \quad\left(A_{0}, A_{1}, \quad, A_{k}\right) M^{(k+1)}(a)
\end{aligned}
$$

Comparing these two results we obtain that $M^{(k+1)}(a)=A_{k+1}$, which is enough to conclude that $\Delta=0$.

In the other cases the proof is similar, except in the case $s=0$ which is left to the reader.

## 3. Applications

Now we will restrict our attention to the case when $r=0$ and the $x_{i}$ are power functions.

The case when $n=1$.
Set: $r=0, n=1, a=0, b=1, x_{i}(t)=t^{a_{i} p_{i}+1}$ in (8), where $a_{i}>-\frac{1}{p_{i}}$ for $i=1, \ldots, m, p_{i}>0$ and $\sum_{i=1}^{\prime \prime \prime} \frac{1}{p_{i}}=1$. We obtain that $\Delta=0$ and

$$
\begin{equation*}
\int_{0}^{1} t^{a_{1}+\cdots+a_{m \prime}} f(t) d t \geq \frac{\prod_{i=1}^{\prime \prime \prime}\left(a_{i} p_{i}+1\right)^{1 / p_{i}}}{1+\sum_{i=1}^{m \prime \prime} a_{i}} \prod_{i=1}^{m}\left(\int_{0}^{1} t^{a_{i} p_{i}} f(t) d t\right)^{1 / p_{i}} \tag{12}
\end{equation*}
$$

if $f$ is a nondecreasing function. It is an improvement of Pólya's inequality (4). Some other results related to this inequality can be found in [5] and [8].
For example, combining (12) and the inequality

$$
\sum_{i=1}^{m} a_{i}+2 \geq \prod_{i=1}^{m}\left(a_{i} p_{i}+2\right)^{1 / p_{i}}
$$

which follows from the inequality between arithmetic and geometric means, we obtain

$$
\begin{align*}
\int_{0}^{1} t^{a_{1}+\cdots+a_{m \prime}} f(t) d t \geq & \frac{\prod_{i=1}^{\prime \prime \prime}\left(\left(a_{i} p_{i}+1\right)\left(a_{i} p_{i}+2\right)\right)^{1 / p_{i}}}{\left(1+\sum_{i=1}^{\prime \prime \prime} a_{i}\right)\left(2+\sum_{i=1}^{\prime \prime \prime} a_{i}\right)} \\
& \times \prod_{i=1}^{m}\left(\int_{0}^{1} t^{a_{i} p_{i}} f(t) d t\right)^{1 / p_{i}} \tag{13}
\end{align*}
$$

The case when $n=2$.
Set: $r=0, n=2, a=0, b=1, x_{i}(t)=t^{a_{i} p_{i}+2}$ in (8), where $a_{i}>-\frac{1}{p_{i}}$ for $i=1, \ldots, m, p_{i}>0$ and $\sum_{i=1}^{\prime \prime \prime} \frac{1}{p_{i}}=1$. After some simple calculation, we obtain that $\Delta=0$ and inequality (13) holds if $f$ is a concave function. So inequality (13) applies not only for $f$ nondecreasing, but also for $f$ concave.

## 4. Results for Quasiarithmetic Means

Definition 2. Let $f$ be a monotone real function with inverse $f^{-1}, p=$ $\left(p_{1}, \ldots, p_{n}\right)=\left(p_{i}\right)_{i}, a=\left(a_{1}, \ldots, a_{n}\right)=\left(a_{i}\right)_{i}$ be real $n$-tuples. The quasiarithmetic mean of $n$-tuple $a$ is defined by

$$
M_{f}(a ; p)=f^{-1}\left(\frac{1}{P_{n}} \sum_{i=1}^{n} p_{i} f\left(a_{i}\right)\right)
$$

where $P_{n}=\sum_{i=1}^{n} p_{i}$.
For $p_{i} \geq 0, P_{n}=1, f(x)=x^{r}(r \neq 0)$ and $f(x)=\ln x(r=0)$ the quasiarithmetic mean $M_{f}(a ; p)$ is the weighted mean $M_{p}^{[r]}(a)$ of order $r$.
Theorem 3. Let $p$ be a positive $n$-tuple, $x_{i} \quad[a, b] \rightarrow \mathbf{R}(i=1, ., n)$ be nonnegative functions with continuous first derivative such that $x_{i}(a)=x_{j}(a), x_{i}(b)=$ $x_{j}(b), i, j=1, \ldots, n$
a) If $\varphi$ is a nonnegative nondecreasing function on $[a, b]$ and if $f$ and $g$ are convex increasing or concave decreasing functions, then

$$
\begin{equation*}
M_{f}\left(\left(\int_{a}^{b} x_{i}^{\prime}(t) \varphi(t) d t\right)_{i} ; p\right) \geq \int_{a}^{b} M_{g}^{\prime}\left(\left(x_{i}(t)\right)_{i} ; p\right) \varphi(t) d t \tag{14}
\end{equation*}
$$

If $f$ and $g$ are concave increasing or convex decreasing functions, the inequality is reversed.
b) If $\varphi$ is a nonnegative nonincreasing function on $[a, b]$, fconvex increasing or concave decreasing function and $g$ is concave increasing or convex decreasing, then (14) bolds.

If $f$ is concave increasing or convex decreasing function and $g$ is convex increasing or concave decreasing, then (14) is reversed.

Proof: Suppose that $\varphi$ is nondecreasing and $f$ and $g$ are convex functions. We shall use integration by parts and the well-known Jensen inequality for convex functions. The latter states that if $\left(p_{i}\right)$ is a positive $n$-tuple and $a_{i} \in I$, then for every convex function $f \quad I \rightarrow \mathrm{R}$ we have

$$
\begin{equation*}
f\left(\frac{1}{P_{n}} \sum_{i=1}^{n} p_{i} a_{i}\right) \leq \frac{1}{P_{n}} \sum_{i=1}^{n} p_{i} f\left(a_{i}\right) \tag{15}
\end{equation*}
$$

We have

$$
\begin{aligned}
M_{f} & \left(\left(\int_{a}^{b} x_{i}^{\prime}(t) \varphi(t) d t\right)_{i} ; p\right)=f^{-1}\left(\frac{1}{P_{n}} \sum_{i=1}^{n} p_{i} f\left(\int_{a}^{b} x_{i}(t) \varphi(t) d t\right)\right) \\
\geq & \frac{1}{P_{n}} \sum_{i=1}^{n} p_{i} \int_{a}^{b} x_{i}^{\prime}(t) \varphi(t) d t=\int_{a}^{b} \frac{1}{P_{n}}\left(\sum_{i=1}^{n} p_{i} x_{i}^{\prime}(t)\right) \varphi(t) d t \\
= & \left.\frac{1}{P_{n}} \sum_{i=1}^{n} p_{i} x_{i}(t) \varphi(t)\right|_{a} ^{b}-\int_{a}^{b} \frac{1}{P_{n}}\left(\sum_{i=1}^{n} p_{i} x_{i}(t)\right) d \varphi(t) \\
\geq & \left.\frac{1}{P_{n}} \sum_{i=1}^{n} p_{i} x_{i}(t) \varphi(t)\right|_{a} ^{b}-\int_{a}^{b} g^{-1}\left(\frac{1}{P_{n}}\left(\sum_{i=1}^{n} p_{i} g\left(x_{i}(t)\right)\right) d \varphi(t)\right. \\
= & \left.\left.\frac{1}{P_{n}} \sum_{i=1}^{n} p_{i} x_{i}(t) \varphi(t)\right|_{a} ^{b}-\int_{a}^{b} M_{g}\left(x_{i}(t)\right)_{i} ; p\right) d \varphi(t) \\
= & \left.\frac{1}{P_{n}} \sum_{i=1}^{n} p_{i} x_{i}(t) \varphi(t)\right|_{a} ^{b}-\left.M_{g}\left(\left(x_{i}(t)\right)_{i} ; p\right) \varphi(t)\right|_{a} ^{b} \\
& +\int_{a}^{b} M_{g}^{\prime}\left(\left(x_{i}(t)\right)_{i} ; p\right) \varphi(t) d t=\int_{a}^{b} M_{g}^{\prime}\left(\left(x_{i}(t)\right)_{i} ; p\right) \varphi(t) d t .
\end{aligned}
$$

Theorem 4. Let $x_{i}, i=1, \quad, n$, satisfy assumptions of Theorem 4 and let $p$ be a real $n$-tuple sucb that

$$
\begin{equation*}
p_{1}>0, \quad p_{i} \leq 0 \quad(i=2, \quad, n), \quad P_{n}>0 \tag{16}
\end{equation*}
$$

a) If $\varphi$ is a nonnegative nonincreasing function on $[a, b]$ and if $f$ and $g$ are concave increasing or convex decreasing functions, then (14) bolds, while if $f$ and $g$ are convex increasing or concave decreasing (14) is reversed.
b) If $\varphi$ is a nonnegative nondecreasing function on $[a, b], f$ is convex increasing or concave decreasing and $g$ concave increasing or convex decreasing, then (14) bolds.

If f is concave increasing or convex decreasing and $g$ is convex increasing or concave decreasing, then (14) is reversed.

The proof is similar to that of Theorem 4. Instead of Jensen's inequality, a reverse Jensen's inequality [3, p. 6] is used: that is, if $p_{i}$ is real $n$-tuple such that (16) holds, $a_{i} \in I, i=1, \ldots, n$, and $\left(1 / P_{n}\right) \sum_{i=1}^{n} p_{i} a_{i} \in I$, then for every convex function $f \quad I \rightarrow \mathrm{R}(15)$ is reversed.

Remark 3. In Theorem 4 and 5 we deal with first derivatives. We can state an analogous result for higher-order derivatives as in Section 2.

Remark 4. The assumption that $p$ is a positive $n$-tuple in Theorem 4 can be weakened to $p$ being a real $n$-tuple such that

$$
0 \leq \sum_{i=1}^{k} p_{i} \leq P_{n} \quad(1 \leq k \leq n), \quad P_{n}>0
$$

and $\left(\int x_{i}^{\prime}(t) \varphi(t) d t\right)_{i}$ and $\left(x_{i}(t)\right)_{i}, t \in[a, b]$ being monotone $n$-tuples.
In that case, we use Jensen-Steffenen's inequality [3, p. 6]. instead of Jensen's in-equality in the proof.

In Theorem 5 , the assumption on $n$-tuple $p$ can be replaced by $p$ being a real $n$-tuple such that for some $k \in\{1, \ldots, m\}$

$$
\sum_{i=1}^{k} p_{i} \leq 0(k<m) \quad \text { and } \quad \sum_{i=k}^{n} p_{i} \leq 0(k>m)
$$

and $\left(\int x_{i}^{\prime}(t) \varphi(t) d t\right)_{i},\left(x_{i}(t)\right)_{i}, t \in[a, b]$ being monotone $n$-tuples.
We use the reverse Jensen-Steffensen's inequality (see [3, p. 6] and [4]) in the proof.

## 5. Results for Logarithmic Means

We define the logarithmic mean $L_{r}(x, y)$ of distinct positive numbers $x, y$ by

$$
L_{r}(x, y)=\left\{\begin{array}{cc}
\left(\frac{1}{y-x} \frac{y^{r+1}-x^{r+1}}{r+1}\right)^{1 / r} & r \neq-1,0 \\
\frac{1}{e}\left(\frac{y^{y}}{x^{x}}\right)^{\frac{1}{r-x}} & r=0 \\
\frac{\ln y-\ln x}{y-x} & r=-1
\end{array}\right.
$$

and take $L_{r}(x, x)=x$. The function $r \mapsto L_{r}(x, y)$ is nondecreasing.

It is easy to see that $L_{1}(x, y)=\frac{x+y}{2}$ and using method similar to that of the previous theorems we obtain the following result.

Theorem 5. Let $g, b \quad[a, b] \mapsto \mathbf{R}$ be nonnegative nondecreasing functions with continuous first derivatives and $g(a)=b(a), g(b)=b(b)$.
a) Iff is a nonnegative increasing function on $[a, b]$, and if $r, s \leq 1$, then

$$
\begin{equation*}
L_{r}\left(\int_{a}^{b} g^{\prime}(t) f(t) d t, \int_{a}^{b} b^{\prime}(t) f(t) d t\right) \leq \int_{a}^{b} L_{s}^{\prime}(g(t), h(t) f(t) d t \tag{16}
\end{equation*}
$$

If $r, s \geq 1$ then the reverse inequality bolds.
b) Iff is a nonnegative nonincreasing function then for $r<1<s$ (16) bolds, and for $r>1>s$ the reverse inequality bolds.

Proof: Let $f$ be a nonincreasing function and $r<1<s$. Using $F=-f$, integration by parts and inequalities between logarithmic means we get

$$
\begin{aligned}
& L_{r}\left(\int_{a}^{b} g^{\prime}(t) f(t) d t, \int_{a}^{b} b^{\prime}(t) f(t) d t\right) \\
& \leq L_{1}\left(\int_{a}^{b} g^{\prime}(t) f(t) d t \int_{a}^{b} b^{\prime}(t) f(t) d t\right)=\frac{1}{2} \int_{a}^{b}(g(t)+b(t))^{\prime} f(t) d t \\
& =\left.\frac{1}{2}(g(t)+b(t)) f(t)\right|_{a} ^{b}+\int_{a}^{b} \frac{1}{2}(g(t)+b(t)) d F(t) \\
& \leq\left.\frac{1}{2}(g(t)+b(t)) f(t)\right|_{a} ^{b}+\int_{a}^{b} L_{s}(g(t), b(t)) d F(t) \\
& =\left.\frac{1}{2}(g(t)+b(t)) f(t)\right|_{a} ^{b}-\left.L_{s}(g(t), b(t)) f(t)\right|_{a} ^{b} \\
& \quad+\int_{a}^{b} L_{s}^{\prime}(g(t), b(t)) f(t) d t=\int_{a}^{b} L_{s}^{\prime}(g(t), b(t)) f(t) d t
\end{aligned}
$$

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