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An Infinite Set of Solid Packings on the Sphere

By

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Abstract

The simple concept of solid packing and solid covering was introduced by L. Fejes Tóth [5]. In this paper we consider the following example. We place $2n$ congruent non-overlapping circles with their centres at the vertices of the Archimedean tiling $(3, 3, 3, n)$ such that each circle touches 4 others. The system is supplemented by two additional circles, each touching n circles of the system. It will be proved that this packing is solid, for all $n \geq 4$.

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1. Introduction

A set of open (closed) discs is said to form a *packing (covering)* in the Euclidean or hyperbolic plane or on the unit sphere if each point of the plane or the sphere belongs to at most (at least) one disc of the set. The literature on the subject of packing and covering in two and higher dimensions is very extensive; see e.g., [2], [3], [4], [6] and the references given there.

The most frequently employed method of measuring the efficiency of a packing or covering is to determine their density. As usual, the density of a system of sets on the unit sphere S^2 is defined as the ratio of the total area of the sets to the surface area of the sphere. To give

an example, we consider an equilateral triangle of side $2r(r \leq \pi/3)$ and the system of three circles of radius r whose centres lie on the vertices of the triangle. The density of any packing of at least three circles with radius r is not greater than the area of the part of the triangle covered by the three circles, divided by the area of the whole triangle.

A new aspect of packing and covering offered itself when L. Fejes Tóth introduced the notion of solidity (see [5]). In the Euclidean or hyperbolic plane or on the unit sphere, a packing (covering) of discs is said to be *solid* if *no finite* number of its members can be rearranged so as to form, together with the rest of the members, a packing (covering) *not congruent* to the original one. A solid packing of n equal circles on the sphere is always a densest packing of n equal circles, and similarly for coverings. Note, however, that the converse is not true. Solid sets of circles were investigated by several authors (see, e.g. [1], [5], [7], [9], [10], [11]).

In this paper we prove the solidity of an infinite set of packings of circles associated with the Archimedean tilings $(3,3,3,n)$, for any $n \geq 4$. A. Heppes suggested to the author to consider these packings. The proof makes use of some recent results of the author ([8]).

2. Definitions and Notation

Let $r_1 < r_2 < \quad < r_n$ be given positive numbers such that

$$r_1 + r_n \leq \frac{\pi}{2} \tag{1}$$

A circle on S^2 whose radius belongs to the set

$$\mathscr{R} = \{r_1, r_2, \quad , r_n\}$$

will be said to be *admissible*. The packings we will consider, are *saturated*, which means that the packing leaves no free room for an additional circle of admissible size without overlapping. A saturated packing of admissible circles consists of more than three circles and generates a *Delaunay triangulation* of the sphere. The construction of this triangulation is described in detail in a paper by Molnár [12]. The result is an edge-to-edge tiling of S^2 with the following properties:

(i) The faces of the tiling are triangles whose vertices are the centres of the circles,
(ii) each triangle is contained in an open hemisphere, and
(iii) no circle intersects or touches the opposite side of the triangle.

We assign a *positive weight* $w(r_i)$ to the circle C_i of radius r_i

$$r_i \to w(r_i) > 0 \tag{2}$$

and call the product

$$a(C_i)w(r_i) = K(r_i) = K_i \tag{3}$$

the *weighted area* of C_i, where $a(C_i) = 2\pi(1 - \cos r_i)$ is the ordinary area of C_i.

In an open hemisphere, let t be a triangle spanned by the centres O_1, O_2, O_3 of three *non-overlapping* circles C_1, C_2, C_3 with radii r_1, r_2 and r_3, respectively. Let us assume that *none of the circles intersects (but possibly touches) the opposite side of t*. We call such a triple of circles a *normal triple* and the associated triangle t a *normal triangle*. For a normal triple of circles with radii r_1, r_2, r_3 we define the function

$$\delta = \frac{\sum_{i=1}^{3} K_i \alpha_i}{2\pi\Delta} \tag{4}$$

where α_i is the angle of t at O_i, K_i is the weighted area of C_i, and Δ denotes the area of t. We call δ the weighted density of the three circles in t, or in short, *the density in t*.

Let $\mathscr{R} = \{r_1, . \quad , r_n\}$ be a set of radii less than $\pi/2$, and let a positive weight be assigned to each element of $\mathscr{R}$. We denote the set of these assigned weights by $\mathscr{W}$ For a given set $\mathscr{R}$ and an assigned set $\mathscr{W}$, we consider all normal triples consisting of circles with radii from $\mathscr{R}$, and the associated normal triangles. The triangles of maximal density δ will be called *extremal triangles* defined by the sets $\mathscr{R}$ and $\mathscr{W}$ Observe that there can be several types of extremal triangles, as the solution of this problem is possibly not unique.

A normal triangle t is said to be *tight* if the three circles touch each other. A normal triangle is said to be *stretched* if one of the circles touches the two others and the opposite side of the triangle t. If the three circles with radii r_1, r_2, r_3 touch each other, then the weighted density in t will be denoted by

$$D(r_1, r_2, r_3).$$

In the next section we shall use the concept of weighted density to prove the solidity of a specified set of packings of circles.

3. A Solid Packing of Circles Associated with $(3, 3, 3, n)$

The Archimedean tiling $(3, 3, 3, n)$ consists of $2n$ congruent equilateral triangles and 2 congruent regular n-gons, three triangles and

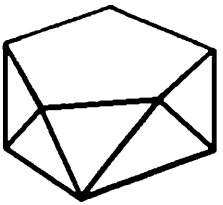

Fig. 1

an n-gon meeting at each vertex. The corresponding polyhedron is a regular n-sided antiprism with all edges of the same length (see Fig. 1).

Let $2r$ denote the edge-length of the tiling. We consider the packing $\mathscr{P}'$ of circles with radius r centred at the vertices, so that each circle of $\mathscr{P}'$ touches four others. For $n = 3$ and 4, this arrangement is solid, since it forms the unique densest packing of 6 and 8 equal circles, respectively (see [6], pp. 114, 164). For $n \geq 5$, however, the packing $\mathscr{P}'$ is definitely non-solid: Two circles with radius r can be placed in the n-gons of the tiling without overlapping a circle of $\mathscr{P}'$ This suggests to add to $\mathscr{P}'$ two circles of radius R, say, concentric with the n-gons of the tiling, each touching n circles of $\mathscr{P}'$ We denote this supplemented packing by $\mathscr{P}$.

Theorem 1. *The packing $\mathscr{P}$ is solid, for all $n \geq 4$.*

Proof: We decompose both n-gons of $(3,3,3,n)$ into n congruent isosceles triangles with the angle $\frac{2\pi}{n}$ at the apex. This way, we get the Delaunay tiling generated by $\mathscr{P}$ consisting of $2n$ congruent isosceles triangles T_i and $2n$ congruent equilateral triangles T_e, with two adjacent triangles T_i and three of type T_e around each vertex of $(3,3,3,n)$. Let α be the interior angle of T_e and $\beta/2$ the angles at the base of T_i. Then we have

$$\begin{aligned} \sin\frac{\alpha}{2} &= \frac{1}{2\cos r}, \\ \cos\frac{\pi}{n} &= \cos r \sin\frac{\beta}{2}, \\ 3\alpha + \beta &= 2\pi \end{aligned} \tag{5}$$

and, as a consequence,

$$2\cos\alpha + 1 = 2\cos\frac{\pi}{n}. \tag{6}$$

From

$$\cos(R+r) = \cot\frac{\beta}{2}\cot\frac{\pi}{n} = -\cot\frac{3\alpha}{2}\cot\frac{\pi}{n} \tag{7}$$

in conjunction with (6) we obtain

$$\cos(R+r) = \sqrt{\frac{(1-2\cos\alpha)(1+\cos\alpha)}{(3+2\cos\alpha)(1-\cos\alpha)}}. \tag{8}$$

Because $\alpha > \pi/3$, we see from (7) that

$$R + r < \frac{\pi}{2}. \tag{9}$$

Observe that

$$\tan(R+r) = \sqrt{\frac{2}{(1-2\cos\alpha)(1+\cos\alpha)}} \tag{10}$$

is a strictly decreasing function of α. If n is given, then α, r and R can be calculated from (6), (5) and (7) (or (10)).

Lemma. *The parameters α and r are strictly decreasing functions, and $R + r$ and R are strictly increasing functions of n. Furthermore*

$$\lim \alpha = \frac{\pi}{3}, \quad \lim r = 0, \quad \lim R = \frac{\pi}{2} \tag{11}$$

as $n \to \infty$.

The proof is obvious.

For $n = 4$, the packing $\mathscr{P}'$ is solid. Two additional circles of radius R must have their centres at the midpoints of the quadrangles of $(3,3,3,4)$. Thus $\mathscr{P}$ is solid as well.

For $n = 5$, we find from (6), (5) and (7) that $\alpha = \frac{2\pi}{5}$ and $r = R = \arccos(1/(2\sin\frac{\pi}{5}))$ which is the inradius of the regular tiling $\{5,3\}$. Because the incircles of $\{5,3\}$ form the only densest packing of 12 congruent circles, the packing $\mathscr{P}$ is solid for $n = 5$.

In order to prove the solidity of $\mathscr{P}$ for $n \geq 6$, we make use of two theorems established in [8].

Theorem 2 (Theorem 1 in [8]). *Let $r_1 < r_2 < \quad < r_n$ be positive numbers less than $\pi/2$. A positive weight $w(r_i)$ is assigned to circles of radius r_i, for $i = 1, \quad , n$. If $r_i < r_j$ we assume that the weighted areas satisfy*

$$K(r_i) < K(r_j) \quad (r_i < r_j). \tag{12}$$

Consider a normal triple of circles with radii from $\{r_1, \ldots, r_n\}$, *and let t be the corresponding normal triangle. Then the weighted density in t satisfies*

$$\delta \leq \max_{1 \leq i \leq j \leq k \leq n} D(r_i, r_j, r_k) \equiv S(r_1, \ldots, r_n). \tag{13}$$

In the case of equality, the triangle t is either tight or stretched.

Theorem 3 (Theorem 3 in [8]). *Consider a saturated packing of circles with radii from* $\{r_1, \ldots, r_n\}$, *where* $r_1 + r_n \leq \pi/2$. *Let us assume that a positive weight can be assigned to each radius such that the following three conditions are satisfied*:

(i) *If* $r_i < r_j$, *the weighted areas satisfy*

$$K(r_i) < K(r_j); \tag{14}$$

(ii) *the Delaunay decomposition of the packing consists of extremal triangles*;

(iii) *there is (up to congruence) only one edge-to-edge tiling composed of extremal tight triangles such that the associated sectors fit together to form complete circles.*

Then the packing is solid.

Theorem 2 states that $S(r_1, \ldots, r_n)$ represents the density in an extremal triangle.

The packing $\mathscr{P}$ consists of circles with two radii $r < R$, where $R + r < \pi/2$ (see (9)). The tiling generated by $\mathscr{P}$ is composed of two types of tight triangles, T_e and T_i, associated with the triple (r, r, r) and (r, r, R), respectively. We assign to circles of radius r the weight $w(r) = 1$, and to circles of radius R such a weight $w(R) = w$ that

$$D(r, r, r) = D(r, r, R). \tag{15}$$

For the weighted densities we find

$$D(r, r, r) = \frac{3\alpha(1 - \cos r)}{3\alpha - \pi} \tag{16}$$

and

$$D(r, r, R) = \frac{(2\pi - 3\alpha)(1 - \cos r) + \frac{2\pi}{n}(1 - \cos R)w}{\frac{2\pi}{n} - (3\alpha - \pi)}.$$

It is easy to show that condition (15) is equivalent to

$$\left(\frac{3\alpha}{3\alpha - \pi} - n\right)(1 - \cos r) = (1 - \cos R)w. \tag{17}$$

We proceed to prove that for $n \geq 6$

$$\frac{3\alpha}{3\alpha - \pi} - n > 1 \tag{18}$$

or, equivalently,

$$3\alpha - \pi < \frac{\pi}{n}. \tag{19}$$

By (6), inequality (19) takes the form

$$\cos\alpha + \tfrac{1}{2} < \cos(3\alpha - \pi)$$

or

$$4\cos^3\alpha - 2\cos\alpha + \tfrac{1}{2} < 0. \tag{20}$$

The left side can be written as a product

$$\left(\cos\alpha - \tfrac{1}{2}\right)\left(4\cos^2\alpha + 2\cos\alpha - 1\right),$$

where the first factor is negative, as $\alpha > \pi/3$, and the second factor is positive, as $\alpha < 2\pi/5$. Thus (20) and (18) are proved. From (18) one sees that the weight w calculated by means of (17) satisfies the requirement (14) on the weighted areas.

The density in an extremal triangle is given by

$$S(r,R) = \max\{D(r,r,r), D(r,r,R), D(r,R,R), D(R,R,R)\}.$$

Condition (ii) of Theorem 3 will be satisfied if we can show that for $n \geq 6$

$$D(r,r,r) > \max\{D(r,R,R), D(R,R,R)\}. \tag{21}$$

The values of $D(r,r,r)$ and $D(r,R,R)$ in Table 1 are calculated from (16) and formulae (23) and (24) below (values indicated are truncated to six decimal places).

Using (5), $D(r,r,r)$, as given by (16), can be expressed in terms of α. It can be shown that D is a strictly decreasing function of α for $\pi/3 < \alpha \leq \pi$. Thus $D(r,r,r)$ is a strictly decreasing function of r for $0 < r \leq \pi/3$. Table 1 shows that $w(R) < 1$ for $6 \leq n \leq 10$ (and at least for $n \leq 19$). Taken together, these two facts imply that for $n \leq 10$

$$D(r,r,r) > D(R,R,R). \tag{22}$$

As $R > \pi/3$ when $n > 10$, the configuration (R,R,R) does not exist in this case.

Table 1

n	r	R	$w(R)$	$D(r,r,r)$	$D(r,R,R)$
6	0.477658	0.688898	0.998549	0.899279	0.895250
7	0.418723	0.795515	0.988075	0.901243	0.891481
8	0.372012	0.880738	0.977712	0.902537	0.888267
9	0.334264	0.949982	0.968924	0.903435	0.886059
10	0.303220	1.007132	0.961707	0.904083	0.884702
11	0.277296	1.054982	0.955792	0.904565	0.883969
12	0.255353	1.095560	0.950909	0.904934	0.883672
13	0.236559	1.130365	0.946836	0.905222	0.883673
14	0.220294	1.160520	0.943402	0.905451	0.883873
15	0.206088	1.186880	0.940476	0.905636	0.884205
16	0.193579	1.210110	0.937961	0.905788	0.884622
17	0.182483	1.230726	0.935778	0.905914	0.885090
18	0.172577	1.249141	0.933868	0.906020	0.885589
19	0.163681	1.265685	0.932185	0.906110	0.886102

We now consider three mutually touching circles with radii r, R, R and the triangle determined by their centres. Let γ denote the angles at the base and δ the angle at the apex. Then we have

$$\cos\gamma = \cot(R+r)\tan R, \tag{23}$$

$$\sin R = \sin(R+r)\sin\frac{\delta}{2}$$

and

$$D(r,R,R) = \frac{2\gamma(1-\cos R)w + \delta(1-\cos r)}{2\gamma+\delta-\pi}. \tag{24}$$

One takes from Table 1 that for $6 \le n \le 19$

$$D(r,r,r) > D(r,R,R). \tag{25}$$

Making use of (23) and (24) it is easy to show that

$$\lim D(r,R,R) = \frac{\pi}{\sqrt{12}} = \lim D(r,r,r),$$

as $n \to \infty$. In the following we shall prove that (25) remains true for all $n \ge 20$.

By (17), the term $(1-\cos R)w$ in (24) can be expressed by $1-\cos r$. Referring to (16) and making some elementary transformations one finds that (25) is equivalent to the inequality

$$(3\alpha-\delta)\frac{\pi}{n} - 6\gamma\left(\alpha - \frac{\pi}{3}\right) < 0. \tag{26}$$

In view of (11) and (23) we have

$$\lim \delta = \pi$$

as $n \to \infty$. Therefore, both terms on the left side of (26) tend to zero, as $n \to \infty$. This fact suggests to examine, instead of (26), the equivalent inequality

$$\frac{3\alpha - \delta}{\sin\frac{\pi}{n}} \frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}} - 6\gamma \frac{\alpha - \frac{\pi}{3}}{\sin^2\frac{\pi}{n}} < 0. \tag{27}$$

Let $n_0 \geq 6$ be a given integer, and let $\alpha_0 < \pi/2$ be such that

$$2\cos\alpha_0 + 1 = 2\cos\frac{\pi}{n_0}. \tag{28}$$

From (6) and (28) it follows that for $n \geq n_0$

$$\frac{\pi}{3} < \alpha \leq \alpha_0, \tag{29}$$

which we shall use repeatedly. Our object is to find to the left side of (27) an upper bound depending only on n_0 and α_0.

Considering the second term in (27) and writing $(R + r) - r$ for R we obtain

$$\cot(R + r)\tan R = \frac{1 - \tan r \cot(R + r)}{1 + \tan r \tan(R + r)}. \tag{30}$$

From (5) it follows that

$$\tan r = \sqrt{1 - 2\cos\alpha}. \tag{31}$$

The combination of (10), (23), (30) and (31) yields

$$\cos\gamma = \frac{1 - (1 - 2\cos\alpha)\sqrt{(1 + \cos\alpha)/2}}{1 + \sqrt{2/(1 + \cos\alpha)}}. \tag{32}$$

Because the numerator is strictly decreasing and the denominator is strictly increasing in α, and $\alpha > \pi/3$, one concludes that

$$\gamma > \arccos\frac{1}{1 + 2/\sqrt{3}} = \arccos(2\sqrt{3} - 3). \tag{33}$$

From (6) we get

$$\sin^2\frac{\pi}{n} = 1 - \cos^2\frac{\pi}{n} = \left(\frac{3}{2} + \cos\alpha\right)\left(\frac{1}{2} - \cos\alpha\right), \tag{34}$$

whence

$$\sin^2\frac{\pi}{n} = (3+2\cos\alpha)\sin\frac{\alpha+\frac{\pi}{3}}{2}\sin\frac{\alpha-\frac{\pi}{3}}{2}. \tag{35}$$

Hence

$$\frac{\alpha-\frac{\pi}{3}}{\sin^2\frac{\pi}{n}} > \frac{2}{(3+2\cos\alpha)\sin\frac{\alpha+\frac{\pi}{3}}{2}} > \frac{1}{2\sin\frac{\alpha+\frac{\pi}{3}}{2}}. \tag{36}$$

Combining (29), (33) and (36) we finally get

$$6\gamma\frac{\alpha-\frac{\pi}{3}}{\sin^2\frac{\pi}{n}} > \frac{3}{\sin\frac{\alpha_0+\frac{\pi}{3}}{2}}\arccos(2\sqrt{3}-3) \equiv A, \tag{37}$$

for all $n \geq n_0$.

Turning to the first term in (27), it will be convenient to write

$$\frac{3\alpha-\delta}{\sin\frac{\pi}{n}} = \frac{3\alpha-\pi}{\sin\frac{\pi}{n}} + \frac{\pi-\delta}{\sin\frac{\pi}{n}}. \tag{38}$$

From (29) and (35) in conjunction with the fact that $\frac{x}{\sqrt{\sin x}}$ is strictly increasing on $(0,\pi/2)$ we obtain

$$\frac{3\alpha-\pi}{\sin\frac{\pi}{n}} < \frac{6\sqrt{2}}{\sqrt{(3+2\cos\alpha_0)\sqrt{3}}}\frac{(\alpha_0-\frac{\pi}{3})/2}{\sqrt{\sin((\alpha_0-\frac{\pi}{3})/2)}} \equiv B, \tag{39}$$

for all $n \geq n_0$.

Starting from (23) we have

$$\frac{\pi-\delta}{2} = \frac{\pi}{2} - \arcsin\frac{\sin R}{\sin(R+r)} = \arcsin\sqrt{1-\frac{\sin^2 R}{\sin^2(R+r)}}$$
$$= \arcsin\sqrt{\frac{\cot^2 R - \cot^2(R+r)}{1+\cot^2 R}},$$

where

$$\cot R = \cot((R+r)-r) = \frac{\cot(R+r)\cot r + 1}{\cot r - \cot(R+r)}.$$

After some calculation we find

$$\frac{\pi-\delta}{2} = \arcsin\sqrt{\frac{F}{\cot^2 r + 1}}, \tag{40}$$

where

$$F = -\cot^2(R+r) + 2\cot(R+r)\cot r + 1.$$

Using (10) and (31), F can be expressed by

$$F = \frac{2\cos^2\alpha + \cos\alpha + 1}{2} + \sqrt{2(\cos\alpha + 1)}, \tag{41}$$

which is a strictly decreasing function of α. Since

$$\frac{1}{\cot^2 r + 1} = \frac{1 - 2\cos\alpha}{2 - 2\cos\alpha} \tag{42}$$

and $\alpha > \pi/3$, from (40), (41) and (42) we obtain

$$\frac{\pi - \delta}{2} < \arcsin\sqrt{(1+\sqrt{3})(1 - 2\cos\alpha)}. \tag{43}$$

In view of

$$1 - 2\cos\alpha \le 1 - 2\cos\alpha_0 = 2 - 2\cos\frac{\pi}{n_0} \le 2 - 2\cos\frac{\pi}{6} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

the term under the square root is less than 1. To find a more convenient bound to $(\pi - \delta)/2$ we observe that

$$f(x) = \frac{\arcsin x}{x} \tag{44}$$

is strictly increasing on $(0, 1)$. As $\alpha \le \alpha_0$, from (43) we infer that

$$\frac{\pi - \delta}{2} < f\left(\sqrt{(1+\sqrt{3})(1 - 2\cos\alpha_0)}\right)\sqrt{(1+\sqrt{3})(1 - 2\cos\alpha)}. \tag{45}$$

The combination of (45) and (34) yields

$$\frac{\pi - \delta}{\sin\frac{\pi}{n}} < \frac{4\sqrt{1+\sqrt{3}}}{\sqrt{3 + 2\cos\alpha_0}} f\left(\sqrt{(1+\sqrt{3})(1 - 2\cos\alpha_0)}\right) \equiv C \tag{46}$$

for all $n \ge n_0$.

From (38) we conclude that

$$\frac{3\alpha - \delta}{\sin\frac{\pi}{n}} < B + C, \tag{47}$$

where B, C and f are defined by (39), (46) and (44). Combining (37) and (47) we finally obtain for $n \geq n_0$ that

$$\frac{3\alpha - \delta}{\sin\frac{\pi}{n}} \frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}} - 6\gamma \frac{\alpha - \frac{\pi}{3}}{\sin^2\frac{\pi}{n}} < (B + C) \frac{\frac{\pi}{n_0}}{\sin\frac{\pi}{n_0}} - A. \tag{48}$$

Taking $n_0 = 20$, we obtain by (28)

$$\alpha_0 = 1.06135 \qquad < 1.064.$$

Using (37), (39) and (46) one finds

$$(B + C) \frac{\frac{\pi}{20}}{\sin\frac{\pi}{20}} - A < -0.07604\,. \quad < 0. \tag{49}$$

In view of (48) this shows that (27) and (26) are true for $n \geq 20$. Hence (25) holds for $n \geq 20$ als well. Summarizing we can state that condition (ii) of Theorem 3 is satisfied for all $n \geq 6$.

Let us now turn to condition (iii) of Theorem 3. We have proved that T_e (equilateral) and T_i (isosceles) are the only types of extremal tight triangles. We have

$$\frac{2\pi}{6} < \alpha < \frac{2\pi}{5} \tag{50}$$

for the angle α of T_e. Thus an edge-to-edge tiling consisting of copies of T_e and T_i must contain triangles of type T_i, n of them fitting together to form a regular n-gon. The angle at the base of T_i is $\pi - 3\alpha/2$. Referring to (50), it is easy to show that the equation

$$k\alpha + l(2\pi - 3\alpha) = 2\pi$$

has no other solution in non-negative integers k, l than $k = 3$, $l = 1$. Hence 2 adjacent copies of T_i and 3 copies of T_e meet at each base vertex of T_i. This implies that the only tiling satisfying condition (iii) is congruent to the Delaunay triangulation generated by the original packing $\mathscr{P}$ This completes the proof of Theorem 1. □

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