Wulfenia **9** (2002): 1–7

Wulfenia

Mitteilungen des
Kärntner Botanikzentrums
Klagenfurt

Fractals, self-similarity & structures

Dmitry D. Sokoloff

Summary: We present a critical discussion of a quite new mathematical theory, namely fractal geometry, to isolate its possible applications to plant morphology and plant systematics. In particular, fractal geometry deals with sets with ill-defined numbers of elements. We believe that this concept could be useful to describe biodiversity in some groups that have a complicated taxonomical structure.

Zusammenfassung: In dieser Arbeit präsentieren wir eine kritische Diskussion einer völlig neuen mathematischen Theorie, der fraktalen Geometrie, um mögliche Anwendungen in der Pflanzenmorphologie und Planzensystematik aufzuzeigen. Fraktale Geometrie behandelt insbesondere Reihen mit ungenügend definierten Anzahlen von Elementen. Wir meinen, dass dieses Konzept in einigen Gruppen mit komplizierter taxonomischer Struktur zur Beschreibung der Biodiversität verwendbar ist.

Keywords: mathematical theory, fractal geometry, self-similarity, plant morphology, plant systematics

Critical editions of Dean Swift's Gulliver's Travels (see e.g. SWIFT 1926) recognize a precise scale invariance with a factor 12 between the world of Lilliputians, our world and that one of Brobdingnag's giants. Swift sarcastically followed the development of contemporary science and possibly knew that even in the previous century GALILEO (1953) noted that the physical laws are not scale invariant. In fact, the mass of a body is proportional to L^3, where L is the size of the body, whilst its skeletal rigidity is proportional to L^2. Correspondingly, giant's skeleton would be $12^2=144$ times less rigid than that of a Lilliputian and would be destroyed by its own weight if L were large enough (cf. e.g. GALILEO 1953). Cleverly Swift elsewhere recognizes, however, another option for the spatial structure of the association of living beings:

"The vermin only teaze and pinch
Their foes superior by an inch
So, nat'ralists observe, a flea
Hath smaller fleas that on him prey,
And these have smaller fleas to bite 'em,
And so proceed ad infinitum."

(On Poetry: A rhapsody. See SWIFT (1958); original spelling of XVII century is given)

Such a structure containing many similar structural levels with diminishing scale is described as *self-similar*. Further development of the concept of self-similarity has now resulted in a quite well elaborated theory referred to as *fractal geometry*. Now, as in the 18th century, this concept looks like a promising concept to understand the properties of living beings, and to originate a new quantitative method for plant morphology and plant systematics. Below we present a critical discussion of the real possibilities to apply the fractal geometry in a botanical context.

We find that these possibilities are rather limited but, however, may form a reasonable basis for botanical applications.

Properties of self-similar objects are quite different to those of smooth curves, surfaces and other conventional geometrical figures. Mathematicians began investigation of self-similar objects in the middle of the 19th century and the corresponding results are widely presented in mathematical textbooks. However, due to a characteristic but bad habit of mathematicians, these results are usually presented as exotic examples of curves without length, surfaces without area and other peculiar members of the mathematical zoo.

H. MINKOWSKI, who is widely appreciated as a founder of mathematical language for relativity theory, made an important step towards a quantitative description of self-similar objects (MINKOWSKI 1901). He suggested the following general definition for the length of a curve, area of a surface and volume of a body. Let A be a set in 3D space. Surround each point of this set by a sphere of small radius ε. A set A_ε consisting of these balls is referred as a ε-neighbourhood of the set A. Let us now calculate the volume $V(\varepsilon)$ of the set A_ε. It is quite easy to demonstrate that $V(\varepsilon) \approx N\varepsilon^3$ provided that A consists of N points. For a curve with length L, one obtains $V(\varepsilon) \approx 2\pi L\varepsilon^2$. For a surface with area S, we have $V(\varepsilon) \approx 2S\varepsilon$, and for a body with volume V we obtain $V(\varepsilon) \approx V\varepsilon^0$.

MINKOWSKI suggested that these relations could be considered as definitions of length, area and volume. A careful analysis demonstrates that MINKOWSKI's definitions differ slightly from the corresponding canonical definitions from the textbooks of mathematical calculus; nevertheless they are quite adequate for many specific geometrical problems.

Calculus textbooks tell us that there are curves without length, surfaces without area, and bodies without volume (one can also say that the concepts of length, area and volume are ill posed for such objects). A great mathematician, F. HAUSDORFF, stressed in 1918 that the standard examples of such exotic objects possess a scaling

$$V(\varepsilon) \approx M\varepsilon^\alpha, \qquad (1)$$

however α is neither 3, as for a point, nor 2, as for a line, nor 1, as for a surface, nor 0, as for a body. He suggested a definition of *fractal* dimension,

$$\dim A = 3 - \alpha, \qquad (2)$$

and considered M as a measure, i.e. a generalisation of length, area and volume. Grateful contemporary workers refer to these as Hausdorff (fractal) dimension and Hausdorff measure. Hausdorff measure is measured in $cm^{\dim A}$ and coincides with MINKOWSKI's length; area or volume provided that the Hausdorff dimension is integer. Hausdorff dimension can be determined by plotting $V(\varepsilon)$ in $\ln V - \ln \varepsilon$ coordinates. Then the power-low scaling (1) corresponds to a straight line on the plot, its slope gives the dimension, and the interception the vertical gives the measure.

According to HAUSDORFF, exotic curves without length, surfaces without area and bodies without volume should be normally considered as objects with a fractal dimension. An additional fruitful idea concerning fractal sets has been suggested by another well-known mathematician, O. HÖLDER. He demonstrated that the fractal behaviour is usually associated with the lack of differentiability of a function, which gives, say, a fractal curve (HÖLDER 1924). Recall that the derivative of a function f(x) is the limit of the ratio $\Delta f/\Delta x$ for $\Delta x \rightarrow 0$. If this limit does not exists (being, say, infinite), then the function is non-differentiable. However it can then occur that the limit

$$f^{(\mu)}(x)=\lim_{\Delta x \rightarrow 0} \Delta f/\Delta x^{\mu},$$

does exist. Then it is referred as a fractal (Hölder) derivative of order μ. The fractal dimension can be explicitly connected with μ.

The concept of fractal dimension is fruitful for self-similar objects being formally applicable to a wider range of objects. The concept of fractal dimension is tied to the nature of similarity, and GALILEO's arguments concerning the incompatibility of scaling invariance with physical laws fail because the concepts of volume and length are inadequate for the object under consideration.

In the first half of 20th century fractal objects were recognized as things that occur in everyday life, rather than only in mathematical textbooks. Moreover, fractal geometry demonstrated some (specific) importance in practical life. According to the scientific folklore, the story can be presented as follows: A prominent English expert in hydromechanics, Richardson, being a model citizen tried to make a contribution to the defence of his country during the World War I. The government being slightly sceptical concerning his abilities restricted his duties to a problem, which seemed to everybody to be solvable. Based on available maps, Richardson was asked to calculate the length of British coastline, required for defence planning. Taking the problem seriously enough, Richardson recognized that the calculated length of the coastline crucially depends on the scale of the map used, and concluded that the British coastline has no length at all, being a fractal object. The official's reaction on this result was not widely disseminated possibly because of intellectual preconceptions. A more important example of a fractal object is the trajectory of a Brownian particle or, speaking more mathematically, a Wiener process which has a Hölder derivative of order ½ and, correspondingly, a fractal dimension.

Richardson exploited a slightly different definition of fractal dimension to that given by Eqs. (1, 2) (see e.g. ZELDOVICH & SOKOLOFF 1985; MOON 1987). Nowadays mathematicians exploit a wide range of various concepts for the dimension of self-similar objects.

Unfortunately, the prominent mathematicians participating in the development of fractal geometry ignored the necessity to give an accessible and attractive presentation of the theory, and published it in a form unacceptable to the general public. More recently, B. MANDELBROT invented the very word *fractal* and published several books concerning the fractal geometry of nature written in non-technical French and English (MANDELBROT 1975, 1977, 1982, 1988) thus earning popularity greater than that of HAUSDORFF. Obviously, this fact hardly is well accepted by highbrow mathematicians. It illuminates however an important problem of modern times. Modern mathematical ideas now become inaccessible to the general reader, as well as to experts in neighbouring branches of mathematics. Thus, a well-written presentation can open a new epoch of scientific development. We believe that this feature represents an obvious crisis of modern mathematics, connected with its inability to give a wide synthesis of modern mathematical ideas and results comparable to that given by HILBERT at the very beginning of 20^{th} century (cf. SADOVNICHY 2000).

The concept of fractal geometry strongly impressed scientific society, and fractals began to be involved in explanations of various problems important for modern science. In particular, the structure of the lungs allowing a very effective oxygen exchange between blood and air due to its huge surface area and moderate volume became to be referred as fractal. The initial enthusiasm was, however, replaced gradually by a more realistic estimation of the abilities of fractal geometry. The scaling (1) can be valid in any practical situation for a limited range of ε only, and this range is usually not large enough to compel on the fractal nature of the object under consideration (see e.g. AVNIR et al. 1998). The following reason can be even more important. Suppose that careful morphological research gives us a specific value for the fractal dimension of a lung. Why is this result important for biology? How can the conclusion that the British coastline has dimension 1.3 (FEDER 1988) be exploited? Sometimes a result extracted from fractal geometry appears to be useful, say to compare quantitatively a random self-similar object and its computer simulation (e.g. ANUFRIEV & SOKOLOFF 1994); however in many cases the answer is rather unclear.

Standard textbooks of fractal geometry usually avoid examples taken from botany. However naked eye recognition of pictures from an atlas of higher plants (e.g. ROTHMALER 1987) convinces the reader that the border of a leaf can be as meandering as the British coastline and that the fractal dimension could be an adequate indicator of this meandering. Of course, a lot of practical experience with this indicator is needed to be able to decide to what extent it can be useful for purposes of plant systematics. The concept of the pseudocycle can be considered as another example of a fractal object important in botany. This concept focuses attention on the appearance of very similar but non-homological structures of various scales in a sequence of comparable plants (see e.g. KUSNETZOVA 1988). In particular, an inflorescence can become to be very similar to an individual flower, and leads to the development of more and more complicated structural levels, similar to an iterative process leading to construction of fractal objects in mathematics. The numbers of such levels in botany is obviously quite modest and usually does not exceed 3–4, while mathematicians discuss an infinite number of structural levels with gradually diminishing scales, and a physicist would prefer to consider as a fractal an object with at least 10 structural levels. Of course, it is very nice to know that the concept of the pseudocycle can be embedded in the wider scientific context of fractal geometry, however a lot of practical experience is needed to be sure that fractal geometry can give something more than a new language for the pseudocycle concept.

We believe that the following example demonstrate something less trivial concerning a possible role of fractal geometry for botanical studies. Suppose we are investigating a set A, which is initially thought as a collection of isolated points. If we observe this set with better and better resolution and see more and more points contained in it, we conclude that this set is self-similar with nonvanishing fractal dimension. It means implicitly that the question of how many points belong to this set is quite ill posed. Of course, any particular list of points belonging to A would contain a particular number of points, however this number would essentially depend on the resolution with this set is investigated. This situation looks quite familiar to what occurs in higher plant systematics. The answer to the question of how many species of, say, dandelion grow in the European part of Russia strongly depends on the concept of species used. Using finer and finer properties to isolate new species, more and more species in this genus are obtained rather a stabilisation of the species number at some level of accuracy is observed. Possibly, it is better to say that the number of species in this genus is ill defined and to use something like Hausdorff dimension and measure to quantify the situation. Of course, a straightforward application of fractal geometry is constrained by an obvious necessity to take into account other complementary criteria for establishing the existence of a species besides morphological details, e.g. whether a reproductive isolation from other species is stable.

We conclude that the concept of fractal geometry did suggest a new vision for understanding of nature; however the body of specific achievements of this theory is at the moment quite small. Apart from fractal geometry, modern mathematics suggests many other approaches to describe unusual spatial structures which could be interesting for biology. However the level of publicity given to these scenarios cannot be compared to that of fractal geometry. To be specific, we describe below briefly one possible application (ZELDOVICH et al. 1990).

Consider a bacterial population which at the initial instant $t=0$ has a spatial distribution with a density $\varphi_0(x)$ (here x is a spatial coordinate). Let the living conditions of the bacteria be spatially inhomogeneous, so that their reproduction rate can be considered as a Gaussian random value $U(x)$ (more precisely, as a Gaussian random field with a rapid decay of spatial correlations), with zero mean value and dispersion σ^2. If we ignore other processes, the bacteria concentration grow then exponentially in time as

$$\varphi(x, t) = \varphi_0(x)\exp(U(x)t). \qquad (3)$$

At the first sight, the *mean* concentration of bacteria should grow exponentially as well as the concentration itself, and the growth rate is expected to be of order σ. Surprisingly, the mean concentration in fact grows much more rapidly, as

$$<\varphi(x, t)> = \varphi_0(x)\exp(\sigma^2 t^2/2). \qquad (4)$$

In spite of the fact that Eq. (4) can be obtained from Eq. (3) by an explicit evaluation which follows the formal definition of mean concentration, this result strongly violates all qualitative ideas of statistical physics as well as common sense. To resolve the paradox, we note that the Gaussian value U is unbounded and obtains (with a very small probability) values which exceeds the value of σ by an arbitrary large factor. The maximal values of U are located in space at large distances away from each other, and more remote maximums give larger and larger values of the regeneration rate U. The mean concentration at a given moment is generally determined by a very remote maximum of U.

Introducing one more effect, i.e. bacteria diffusion with diffusivity ν, we arrive at an evolution governed by the following equation

$$\partial\varphi/\partial t = U\varphi + \nu\,\Delta\varphi, \qquad (5)$$

which is quite similar to the famous Schrödinger equation of quantum mechanics. Eq. (5) is well-investigated in mathematical physics.

Investigating Eq. (5), experts initially considered biological language as a reasonable tool to make results more acceptable to a general reader. However experts in microbiology considered the results quite seriously. Moreover, the evolution of φ unpleasantly mimics the behaviour of human beings at, say, the initial epoch of industrial development as presented in the books of F. BRAUDEL (1979, 1986). At the initial stage, an isolated maximum of U produces a region of influence where the concentration φ is determined by diffusion from the maximum. Then regions of influence of various maxima begin to be in contact, and an epoch of concurrence between regions takes place. As a result, a region associated with a more pronounced maximum absorbs the neighbouring regions. The general shape of the evolution is very similar to the well-known scenario where Amsterdam, London and New York became leading centres of world economy.

The example considered can be generalized in various respects. One can consider, say, a reproduction rate which is random in time as well as in space, and more complicated models give a more and more detailed simulation of the behaviour of human being. The similarity is so impressive that the natural question arises: do we really need more than a primitive model like (5) to explain our behaviour?

Acknowledgements

Financial support from RFBR under project 02-01-00297 is acknowledged. I am grateful to D. Moss for useful discussion.

References

ANUFRIEV, A. & SOKOLOFF, D. (1994): Fractal properties of geodynamo models. – Geophys. Astrophys. Fluid Dyn. **74:** 207–223.

AVNIR, D., BIHAM, O., LIDAR, D. & MALCAI, O. (1998): Is the geometry of nature fractal? – Science **279:** 39–40.

BRAUDEL, F. (1979): Civilisation matérielle, économie et capitalisme XV–XVIIIe siècle [3 vols.]. – Paris: Armand Colin.

BRAUDEL, F. (1986): La Dynamique du capitalisme. – Paris: Arthaud.

FEDER, J. (1988): Fractals. – New York: Plenum.

GALILEI, G. (1953): Dialogue concerning the two chief world systems, Ptolemaic and Copernician. – Cambridge: Cambridge Univ. Press.

HAUSDORFF, F. (1918): Dimensionen und äußeres Maß. – Math. Ann. **75**: 157–179.

HÖLDER, O. (1924): Die mathematische Methode. Logisch erkenntnistheoretische Untersuchungen im Gebiete der Mathematik, Mechanik und Physik. – Berlin: Springer.

KUZNETZOVA, T. V. (1988): Angiosperm inflorescences and different types of their structural organization. – Flora **181**: 1–17.

MANDELBROT, B. B. (1975): Les Objets Fractals: Forme, Hasard et Dimension. – Paris: Flammarion.

MANDELBROT, B. B. (1977): Fractals: Form, Chance and Dimension. – San Francisco: W. H. Freeman.

MANDELBROT, B. B. (1982): The Fractal Geometry of Nature. – New York: W. H. Freeman.

MANDELBROT, B. B. (1988): Fractals and Multifractals: Noise, Turbulence and Galaxies. – New York: Springer.

MINKOWSKI, H. (1901): Über die Begriffe Länge, Oberfläche und Volumen. – Jahresber. dtsch. Math.-Ver. **9**: 115–127.

MOON, F. C. (1987): Chaotic Vibrations. – New York: John Willey & Sons.

ROTHMALER, W. (1987): Exkursionsflora. Bd. 3. – Berlin: Volk und Wissen.

SADOVNICHY, V. A. (2000): Mathematical Education, its present state and future development. – Moscow: Moscow University. [In Russian]

SWIFT, J. (1926): Gulliver's Travels. [The text of the first edition edited, with an introduction, bibliography and notes, by Harold Wilkinson.] – London: First Edition Club.

SWIFT, J. (1958): The Poems of Jonathan Swift. Ed. by Harold Williams, vol. II [2nd ed.] – Oxford: Clarendon Press.

ZELDOVICH, Y. B., RUZMAIKIN, A. A., SOKOLOFF, D. D. (1990): The Almighty Chance. – Singapur: World Sci.

ZELDOVICH, Y. B., SOKOLOFF, D. D. (1985): Fractals, similarity, intermediate asymptotics – Sov. Phys. Usp. **28:** 606–616.

Address of the author:

Univ.-Prof. Dr. Dmitry D. Sokoloff
Department of Physics
Moscow State University
119899 Moscow
Russia
E-mail: sokoloff@dds.srcc.msu.su