A GROWTH AND YIELD MODEL FOR THINNED STANDS OF YELLOW-POPLAR (*LIRIODENDRON TULIPIFERA*)

by

Harold E. Burkhart
Bruce R. Knoebel
Department of Forestry
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061, U.S.A.

Donald E. Beck
Southeastern Forest Experiment Station
U.S. Forest Service
Asheville, North Carolina 28802, U.S.A.

A GROWTH AND YIELD MODEL FOR THINNED STANDS OF YELLOW-POPLAR (*LIRIODENDRON TULIPIFERA*)

Harold E. Burkhart, Bruce R. Knoebel and Donald E. Beck*

ABSTRACT

Simultaneous growth and yield equations were computed to relate whole-stand basal area growth and cubic volume growth and yield of thinned stands of yellow-poplar (*Liriodendron tulipifera* L.) to initial age and basal area, site index, and projected age. Data from 141 permanent plots in natural even-aged stands of yellow-poplar in the southern Appalachians (U.S.A.) were used to estimate coefficients of the equations.

Stand tables (numbers of trees per unit area by diameter class) were derived from the whole-stand attributes by solving for parameters in a theoretical diameter distribution model (in this case the Weibull distribution was assumed). The parameter estimates of the Weibull distribution were conditioned such that the basal area per acre computed from the resulting stand table equalled the predicted basal area from the stand-level equation.

The model can be used to provide estimates of basal area, numbers of trees, and volumes on an overall stand basis or by diameter class.

---

*The authors are, respectively, Thomas M. Brooks Professor and Graduate Research Assistant in the Department of Forestry, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061, U.S.A., and Project Leader, Southeastern Forest Experiment Station, U. S. Forest Service, Asheville, North Carolina 28802, U.S.A.

INTRODUCTION

In the eastern United States, yellow-poplar (*Liriodendron tulipifera* L.) is an important commercial species that is often cut for lumber and veneer. Because tree size and quality have an impact on the yields of these products, thinning is an important silvicultural tool in yellow-poplar management. Most stands of yellow-poplar can produce a number of lumber- and veneer-size trees without thinning; however, thinning concentrates growth on the best and largest trees. Reliable estimates of stand growth and yield are needed to determine optimal thinning regimes.

In 1972, Beck and Della-Bianca (1972) published equations for predicting basal-area growth and cubic-foot volume growth and yield in stands thinned to various levels of basal area. Subsequently, they published equations to predict board-foot growth and yield and residual quadratic mean stand diameter growth (Beck and Della-Bianca, 1975). The equations were based on measurements taken five years after the initial thinnings on a series of 141 permanent plots.

Since the initial remeasurement, two additional assessments have been taken at 10 and 15 years after the first thinning. The plots were thinned again at the time of the first 5-year remeasurement, thus stand characteristics and tree quality were somewhat different for the second and third 5-year growth periods as compared to the first period. The objective of the research described here was to develop a growth and yield model to predict the development of yellow-poplar stands given a set of initial conditions, a thinning regime, and a rotation age. Data from all three 5-year growth periods were used in this analysis.

DATA

The data analyzed for this study were collected by the U. S. Forest Service, Southeastern Forest Experiment Station, from 141 circular, 1/4-acre plots established in the Appalachian mountains of North Carolina (93 plots), Virginia (31 plots), and Georgia (17 plots). The plots contained 75 percent or more yellow-poplar in the overstory, were free from insect and disease damage, and showed no evidence of past cutting (Beck and Della-Bianca, 1972).

Each plot was thinned (using low thinning) at the time of installation to obtain a range of basal areas for different site-age combinations. Site index (base age 50 years) was determined for each plot using an equation published by Beck (1962). Volumes and basal areas were computed when the plots were thinned and again after five growing seasons. Heights were calculated by fitting a least squares equation relating height to diameter from measurements taken on every tenth tree. From the equation, heights were obtained for each tree in the plot. Then using existing equations (Beck, 1963, 1964), a volume for each tree was computed. Plot volumes were then determined by summing the individual tree volumes. At the time of initial plot establishment, the stands ranged from 17 to 76 years in age, 74 to 138 feet in site index (base age 50 years), and 44 to 208 square feet per acre in basal area.

MODEL STRUCTURE

Several desirable properties were sought when deriving a growth and yield model for thinned stands of yellow-poplar. In particular, we wanted the equations to exhibit analytic compatibility between growth and yield, invariance for projection length, and numeric equivalency between alternative applications of the equations. These properties are exhibited by the stand-level model published by Beck and Della-Bianca (1972) from analysis of the first 5-year growth period; thus, we used their models as a beginning point. In addition to whole stand volume and basal area, we also wanted to derive stand tables to provide flexibility for evaluating the full range of utilization options. Consequently, another goal was to derive stand tables that are compatible with the whole stand values.

Stand-Level Components

Beck and Della-Bianca (1972) fitted the following models (adapted from Sullivan and Clutter, 1972) for prediction of basal area and cubic volume at some projected age when site index, initial age, and basal area are given:

$$\begin{aligned} \ln Y_2 \quad & b_0 + b_1 S^{-1} + b_2 A_2{}^{-1} + b_3 (A_1/A_2) \ln B_1 \\ & + b_4 (1 - A_1/A_2) + b_5 S (1 - A_1/A_2) \end{aligned} \qquad (1)$$

where: Y_2 stand volume per unit area at some projected age A_2; S = site index; B_1 present basal area per unit area; and A_1 = present age.

When A_2 A_1 A and B_2 B_1 B, equation (1) reduces to the general yield model:

$$\ln Y = b_0 + b_1 S^{-1} + b_2 A^{-1} + b_3 \ln B \qquad (2)$$

The yield projection model (1) was derived by substituting a basal-area projection equation for the basal-area term in the general yield model (2). Therefore, inserting $\ln Y_2$, A_2, and $\ln B_2$ into equation 2 and setting the resulting expression equal to the right side of equation (1) and solving the equality for $\ln B_2$ gives the basal area projection model:

$$\ln B_2 \quad (A_1/A_2)\ln B_1 + (b_4/b_3)(1-A_1/A_2) + (b_5/b_3)S(1-A_1/A_2) \qquad (3)$$

Beck and Della-Bianca used ordinary least squares to estimate the coefficients in (1) and substituted the ratios b_4/b_3 and b_5/b_3 as parameter estimates in the basal-area projection equation (3) to ensure that exact numeric equivalency would result when projecting future volume from (1) and when projecting future basal area from (3) and solving for future volume by substitution of appropriate values into (2).

In our analyses, equation (1) was fitted by ordinary least squares to each of the growth periods and standard F-tests were performed to determine if separate coefficients were needed for each period or if data from some of the periods could be combined. From these tests, we determined that two sets of coefficients were needed -- one for the growth period after one thinning and a second for the growth periods following two thinnings. The second thinning apparently altered stand structure and vigor so that growth relationships were significantly affected.

After determining that separate coefficients were needed for the growth periods following one thinning and following two thinnings, final estimates of the parameters in the volume and basal area projection equations were computed by using a simultaneous fitting procedure. This procedure, applied previously by Burkhart and Sprinz[1] to data from thinned loblolly pine plantations, involves minimizing the loss function:

$$F = \frac{\Sigma(Y_i \quad \hat{Y}_i)^2}{\hat{\sigma}_Y^{\,2}} + \frac{\Sigma(B_i - \hat{B}_i)^2}{\hat{\sigma}_B^{\,2}}$$

where Y_i and $\hat{Y}_i$ are the observed and predicted volume values, B_i and $\hat{B}_i$ are the observed and predicted basal area figures, and $\hat{\sigma}_Y^{\,2}$ and $\hat{\sigma}_B^{\,2}$ are estimates of the variance around regression lines for volume and basal area, respectively. These variance estimates, $\hat{\sigma}_Y^{\,2}$ and $\hat{\sigma}_B^{\,2}$, were computed as the mean square error from ordinary least squares regression fits of equations (1) and (3), respectively. Beginning with coefficient estimates from the ordinary least squares fit of (1), the coefficients of models (1) and (3) were adjusted through an iterative process until F in the loss function was minimized. This process of simultaneously fitting the two models (with the imposed restriction that coefficients in the basal area equation are equal to the appropriate ratios of the volume equation coefficients) results in a system of equations that are compatible and numerically consistent. Different weights could be assigned to the two

---

1/ Burkhart, H. E. and P. T. Sprinz. Cubic volume and basal area projection equations for thinned loblolly pine plantations. Submitted to Forest Science.

components, but we felt that for management decisions involving thinning equal weight should be given to both volume and basal area projection. The simultaneous estimation procedure is also more statistically efficient (in that the basal area growth information is used in the fitting) and produces more stable estimates of the basal area equation coefficients for varying units of measure and merchantability standards in (1) than does the derivation of coefficients in (3) from the least squares fit of (1) (Burkhart and Sprinz).

Given equations for estimating the total stand cubic volume and basal area, the volume of selected portions of the stand (e.g., the veneer log portion) according to specified utilization standards can be estimated. This approach does not allow sufficient flexibility, however, to account for rapidly changing utilization standards. Thus an extremely valuable adjunct to the overall stand values is a stand table (numbers of trees per unit area by diameter class). When computing stand tables it is important that they be logically and consistently related to the overall stand characteristics.

Stand-Table Generation

Stand tables were derived from the whole-stand attributes by solving for parameters in a theoretical diameter distribution model (in this case the Weibull distribution was assumed). (Additional background information on the techniques involved in this basic approach can be found in

Strub and Burkhart, 1975; Hyink, 1980; Frazier, 1981; Cao et al, 1982; and Matney and Sullivan, 1982.) Basically, the method of moments was applied to estimate the parameters in the Weibull probability density function (pdf). The equation for the i^{th} non-central moment of X is given by:

$$E(X^i) = \int x^i \, f(x_i, \underline{\theta}) \, dx$$

In the case of forest diameter distributions the first two moments are:

$$E(X) \quad \overline{x} = \text{the average diameter of the stand, and}$$

$$E(X^2) \quad \overline{x^2} = BA/\lambda N$$

where λ = 0.005454 when English units are used and BA and N are basal area and numbers of trees per acre, respectively. Hence, the first two moments of the diameter distribution have stand-level interpretations that are meaningful in forestry practice.

Stand average estimates of the first k moments produce a system of k equations with k unknown parameters which can be solved to obtain estimates of the pdf parameters while ensuring compatibility between whole stand and diameter distribution estimates of the stand attributes described by the moment equations.

Because the moment-based system of equations for the 3-parameter Weibull distribution led to convergence problems, the 3-parameter Weibull pdf was reduced to 2 parameters by using the transformation Y   X-a.

That is, the location parameter "a" was set equal to a constant or predicted outside the system of equations. The two equations for the 2-parameter system are

$$\bar{x} = \int_0^\infty x\, f(x;\, b,c)dx = b\, \Gamma(1 + 1/c)$$

$$\overline{x^2} = \int_0^\infty x^2\, f(x;\, b,c)dx = b^2\, \Gamma(1 + 2/c)$$

The estimated variance of the distribution is given by

$$s^2 = \overline{x^2} - \bar{x}^2 = b^2\, [\Gamma(1 + 2/c) - \Gamma^2(1 + 1/c)]$$

and the coefficient of variation (CV) is estimated by

$$CV = \frac{s}{\bar{x}} = \frac{[\Gamma(1 + 2/c) - \Gamma^2(1 + 1/c)]^{1/2}}{\Gamma(1 + 1/c)}$$

Given estimates of $\bar{x}$ and $\overline{x^2}$, the coefficient of variation is a function of "c" alone and iterative techniques for solving one equation with one unknown can be used to obtain a value for "c". With "c" known, "b" is solved from $\bar{x} = b\, \Gamma(1 + 1/c)$, and "a" is estimated with a constant or equation external to this sytem.

When applying the system, the same stand-level basal area equation is applied when deriving diameter distributions as when estimating overall stand basal area in order to ensure compatibility between the two levels of stand detail.

DISCUSSION

The growth and yield model for yellow-poplar discussed here provides stand-level estimates that are analytically compatible, invariant for projection length, and numerically equivalent with alternative applications of the equations. A joint loss function involving both volume and basal area was used to estimate the coefficients in a system of related equations. This joint minimization produced substantial gains in basal area projection abilities while affecting the accuracy and precision of volume projection very little.

As an adjunct to the stand-level equations, compatible stand tables were derived by solving for the parameters of the Weibull distribution from whole-stand attributes. Evaluations with this and other data sets have shown that stand tables produced through this "parameter recovery" technique approximate observed diameter distributions as well as the somewhat more direct technique of predicting pdf parameters from stand attributes.

Keywords: Volume, Mensuration, Modeling

Prepared: September, 1982

Beck, D. E. 1962. Yellow-poplar site index curves. U.S. Forest Service, Research Note SE-180, 2 pp.

Beck, D. E. 1963. Cubic-foot volume tables for yellow-poplar in the southern Appalachians. U.S. Forest Service, Research Note SE-16, 4 pp.

Beck, D. E. 1964. International 1/4-inch board-foot volumes and board-foot/cubic-foot ratios for southern Appalachian yellow-poplar. U.S. Forest Service, Research Note SE-27, 4 pp.

Beck, D. E. and L. Della-Bianca. 1972. Growth and yield of thinned yellow-poplar. U.S. Forest Service, Research Paper SE-101, 20 pp.

Beck, D. E. and L. Della-Bianca. 1975. Board-foot and diameter growth of yellow-poplar after thinning. U.S. Forest Service, Research Paper SE-123, 20 pp.

Cao, Q. V., H. E. Burkhart and R. C. Lemin, Jr. 1982. Diameter distributions and yields of thinned loblolly pine plantations. School of Forestry and Wildlife Resources, Va. Polytech. Inst. and State Univ., FWS-1-82, 62 pp.

Frazier, J. R. 1981. Compatible whole-stand and diameter distribution models for loblolly pine stands. Ph.D. Dissertation, Va. Polytech. Inst. and State Univ., 125 pp.

Hyink, D. M. 1980. Diameter distribution approaches to growth and yield modeling. In: Forecasting Forest Stand Dynamics (Ed. K. M. Brown and F. R. Clarke). School of Forestry, Lakehead University, Thunderbay, Ontario, p. 138-163.

Matney, T. G. and A. D. Sullivan. 1982. Compatible stand and stock tables for thinned and unthinned loblolly pine stands. Forest Sci. 28:161-171.

Strub, M. R. and H. E. Burkhart. 1975. A class-interval-free method for obtaining expected yields from diameter distributions. Forest Sci. 21:67-69.

Sullivan, A. D. and J. L. Clutter. 1972. A simultaneous growth and yield model for loblolly pine. Forest Sci. 18:76-86.