STOCHASTIC DIFFERENTIAL EQUATIONS AS GROWTH MODELS, SOME EXAMPLES AND UNSOLVED PROBLEMS

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ZUSAMMENFASSUNG

In diesem Beitrag wird auf die Arbeiten von Suzuki im allgemeinen und im besonderen auf den bei dieser Tagung gehaltenen Vortrag eingegangen. Besonders hervorgehoben wurde die Bedeutung der Ergebnisse von Suzuki im Fall der stochastischen Beschreibung der gemeinsamen Entwicklung von Durchmesser und Höhe als zweidimensionaler Vektor. Es besteht ein Bedarf an solchem Instrumentarium auch zur Fortschreibung der Bestandesschaftform. Im zweiten Teil dieser Arbeit wird mit Hilfe der Datenanalyse und aufgrund der speziellen, aber realitätsnahen Modellvorstellungen die Frage der Diffusionseigenschaft beim Wachstumsprozeß des verbleibenden Bestandes untersucht. Die Analyse der Ansätze (9), (10), die aus der stochastischen Differentialgleichung (2) resultieren, ergaben bei umfangreicherem Datenmaterial offensichtliche Widersprüche. Dadurch wurden Korrekturen der Annahmen an den Restprozeß $Y_{t_0,t}$ in (3') und (4) vorgenommen und deren Eigenschaften untersucht.

Keywords: Growth dynamics, Growth prognosis, Stochastic differential equations, Taper curve, Diffusions processes

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INTRODUCTION

It will be possible to arrange the contents of this paper roughly in three parts, which are the following:

1. Tribute will be paid to the wish of the organizer, to the effect that Suzuki's lecture will be dealt with. In doing so, we shall develop some practical connections of these models to real-life forestry problems and to empirical data.

2. In the main chapter, our experiences with so-called diffusion processes used as stand growth models will be put forward. Mainly, a brief methodological discussion of that kind of model will be presented.

3. On the basis of this methodological discussion and of comprehensive data material, we shall suggest some steps toward a gradual improvement of this model basis, especially with regard to predicting the prospective population of even-aged forest stand.

SOME REMARKS CONCERNING SUZUKI'S LECTURE

One of Suzuki's unquestionable merits is the consistent description of stand-growth processes (with regard to crop yield) by means of the analytic theory of stochastic processes. SUZUKI (1971) and SUZUKI-UMEMURA (1974) present models for describing the evolution of a homogeneous, even-aged stand for one dimension BHD. In these works, a calculation $\varphi(t,x)$ of the initial probability density $\varphi(t_o,x)$ with $\varphi(\tau,y) = \int_R \varphi(t_o,x)\cdot p(t,x;\tau,y)dx$ is achieved by means of solving the Kolmogorow-Suzuki partial differential equation $p(t,x;\tau,y)$, whose input functions (drift, variance rate, death rate) characterize the stand. Through a special solution method the death process can be eliminated, which yields an evolution equation for the growth process of the remaining stand. This specific Kolmogorow forward equation describes the so-called diffusion processes. The realisations of such processes are known as Markow processes with continuous paths. For the transition probability or control function, the so-called Chapman-Kolmogorow equation applies, which would imply Markow's property, or loss of memory, of the "stand growth". For this chapter, see SLOBODA (1976, 1977), FROHN (1978) and SLOBODA (1981).

Suzuki's paper (1981) and today's lecture present an organic transfer of this theory onto the stochastic modelling of the two-dimensional growth process of the vector (BHD,HOEHE)~(X_t,Y_t). As a result the Kolmogorow equation for the two-dimensional diffusion process has been deduced, with the latter work concentrating especially on concrete suggestions for two-dimensional drift functions and diffusion functions. These methods are created in analogy with the derivation in the one-dimensional case (SUZUKI (1974)), with the addition of the cross-covariance function, which plays a substantial role in constructing the so-called basic solution of the differential equation. To understand the assumptions that have been made about the covariance functions and cross-covariance functions of the process for (BHD,H), thorough discussions with the author are required for the time being. This holds also for the possible effects of limit transitions carried out with integrands. Provided the above-mentioned difficulties can be dealt with satisfactorily, and if the real process proves to be consistent with Markow's property, the solution thus obtained can be evaluated as a very <u>important means</u> for describing monetary growth. Our results show that the evolution of the overall taper curve can be expressed in the form of a stochastic process. To do this, it suffices to have a knowledge of the two-dimensional growth process (X_t,Y_t) and to make certain additional assumptions (SLOBODA,SABOROWSKI (1981), SLOBODA,SABOROWSKI (1981)). It should be noted, however, that compared to one-dimensional models, it will prove very much more difficult to put this method to a test with a "sufficiently large" amount of data. Reliable measurements of diameters and heights of individual trees growing in long-term experiment areas - measurements which are required by the models in question - have not, to my knowledge, been carried out yet. In the one-dimensional case as well as in the two-dimensional one, questions remain to be answered with regard to estimation and interpretation of the parameters of input functions. In the case of the analytical approach, ML estimation of parameters is recommended (WOLF (1981)).

DIFFUSION PROCESSES AS MODELS FOR DESCRIBING TARGET STAND GROWTH

For describing such processes, there is a purely stochastic analogy to the analytical method. The stochastic differential equation of the Itô type, with certain mathematical restrictions concerning drift function and diffusion function that are irrelevant in the present context, renders an explicit description of the diffusion processes. During research with data from several long-time experimental cultures, the question rightly arose as to whether the stand growth can in fact be regarded as a diffusion process, and if not, what methods are available for correcting the models. The initial reason for asking that sort of question was the infavorable behaviour of model processes with regard to the correlation function R(s,t) of the residual process $Y_{t_o,t}$ (defined below), which is of importance in the description of social behaviour of forest trees in a stand (Fig. 1, Hauersteig 4). This describes rank conservation (maintenance) of trees in the forest stand.

The investigations reveal that for the drift function (direction field for stand growth) the linear function

(1) $\beta(t,X_t) = A(t)X_t+a(t)$

can be presupposed. If we adopt, according to SUZUKI (1974), $\alpha(t,X_t)=B(t)$ (homogeneous state assumption), the stochastic differential equation of the respective diffusion process turns out to be

(2) $dX_t=[A(t)X_t+a(t)]dt+B(t)dW_t$

$X_{t_o} \sim \varphi(t_o,x)$ as initial condition

$W_t \sim$ Wiener-Levy process.

The solution reads

(3) $$X_{t_o,t}=\Phi(t_o,t)\cdot X_{t_o}+\Phi(t_o,t)\int_{t_o}^{t}\Phi^{-1}(t_o,s)a(s)ds+$$
$$+\Phi(t_o,t)\int_{t_o}^{t}\Phi^{-1}(t_o,s)\cdot B(s)dWs$$

i.e.

(3') $X_{t_o,t}=\Phi(t_o,t)\cdot X_{t_o}+c(t_o,t)+Y_{t_o,t}$

where $\Phi(t_o,t)=\exp[\int_{t_o}^{t} A(s)ds]$ and $Y_{t_o,t}$ is a residual process.

If one takes a look at the graphic representation of diameter measurements x_t (y axis) and x_{t_o} (x axis), we detect a linear relation between the momentary diameter x_{t_o} and the future diameter (Fig. 2). In other words, the structure of the prognostic equation (3') can be kept unchanged in its tendency. The method in question is one of linear regression, where $\Phi(t_o,t)$ and $c(t_o,t)$ form the regression coefficients (t_o and t fixed). $Y_{t_o,t}$ represents the error connected with the regression. $Y_{t_o,t}$ in (3') is a stochastic integral and $Y_{t_o,t}$ is stochastically independent on X_{t_o} We obtain following equations for the moments:

(4) $$EY_{t_o,t}=0; \quad \mathrm{Var}\, Y_{t_o,t}=\Psi^2(t_o,t)=\Phi^2(t_o,t)\cdot\int_{t_o}^{t}\frac{B^2(s)}{\Phi^2(t_o,s)}ds.$$

$Y_{t_o,t}$ is a Gaussian-process (SLOBODA (1977,1981)).

In the model, the following holds for the statistical moments of $X_{t_o,t}$

(5) $$EX_{t_o,t}=\Phi(t_o,t)\cdot EX_{t_o}+c(t_o,t)$$

(6) $$\mathrm{Var}\, X_{t_o,t}=\Phi^2(t_o,t)\cdot \mathrm{Var}\, X_{t_o}+\Psi^2(t_o,t)$$

By means of regression estimators, the functions $\hat{\Phi}(t_o,t_i)$, $\hat{c}(t_o,t_i)$, and the residual variance of $Y_{t_o,t}$ with $\hat{\Psi}^2(t_o,t_i)$ for the various points of time $t_i, i=1,\dots,n$ can be estimated. The courses of $\hat{\Phi},\hat{c},\hat{\Psi}^2$ have been exemplified in figures 3a, 3b and 3c. These demonstrate that it is by all means possible to force the regression to run through origin, which means c=0. This procedure would have the advantage that the modelling of $c(t_o,t)$ can be dispensed with, especially since their courses cannot, without further ado, be described by a generally valid analytical model. For the Hauersteig 4 plot, the courses produced by this procedure for $\hat{\Phi}(t_o,t)$, $\hat{\Psi}^2(t_o,t)$ are shown in fig. 4a and 4b. The same tendency can be observed in other even-aged stands.

By reason of data analysis, it seems justified to use the following equations:

(7) $\hat{\Phi}(t_o,t)=\frac{1-Le^{-kt}}{1-Le^{-kt_o}}$ (SLOBODA (1978)) and

(8) $\hat{\Psi}^2(t_o,t)=p\cdot(t-t_o)^q t_o^{\,r}$ (SABOROWSKI (1982)).

Between functions $\Phi(t_o,t)$ and $\Psi(t_o,t)$ and the input functions A(t) and B(t) we can, in the case of the diffusion process, derive the following relations from equation (3):

(9) $[\ln\Phi(t_o,t)]'=A(t)$

(10) $\Psi^{2\prime}(t_o,t)-2\Psi^2(t_o,t)\cdot A(t)=B^2(t)$

According to (4), fitting $\Phi(t_o,t)$ yields $A(t)=\frac{Lke^{-kt}}{1-Le^{-kt}}$.

$\Psi^2(t_o,t)$ and A(t) yield the diffusion coefficient B(t) according to (5). According to (10) and (8) the courses of B(t) depend on t_o, which contradicts the diffusion function in the diffusion processes. The tendency detected her for the course of $\Psi^2(t_o,t)$, whose character seems to be one of general validity, makes it seem justified to think about alternatives with regard to diffusion.

Nevertheless, the theory of stand growth by way of diffusion equations has led to a very simple and workable approximation of growth prediction. Without the devices that the theory of stochastic differential equations places at our disposal, this would not have been possible. It is possible to verify graphically the result in the form of an approximation. The discrepancies between the diffusion process and the method presented above lie in the somewhat differentiated structure of the stochastic time process $Y_{t_o,t}$, and this is where we shall continue with our research work.

CONSEQUENCES FOR FURTHER HANDLING OF SUCH MODELS

The findings presented above are valid not only for parcel 4 of the Hauersteig experimental area, which only served as an example; they have also been verified in 7 other stands, among which are the three remaining Hauersteig parcels:

Stand 1 Bueren ("schwache Hochdurchforstung")
Registration age: 42, 47, 51, 55, 63, 67, 71, 75, 79, 85, 88
Number of stems: 224

Stand 2 Bueren ("schwache Niederdurchforstung")
Registration age: see above, stand 1
Number of stems: 345

Stand 3 Westerhof ("schwache Hochdurchforstung")
Registration age: 37, 41, 45, 49, 56, 60, 63, 67, 71, 75, 79, 81, 85
Number of stems: 93

Stand 4 Westerhof ("schwache Niederdurchforstung")
Registration age: see above, stand 3
Number of stems: 160

Stand 5-8 (Hauersteig parcel 1-4)
Registration age: 37, 40, 44, 48, 51, 56, 59, 64, 71, 76, 81, 84
Number of stems: parcel 1: 137, parcel 2: 163, parcel 3: 162, parcel 4: 138

Summing up, we admit that, strictly speaking, the BHD development in a stand cannot be seen as a diffusion process with drift and diffusion functions of type $A(t)X_t$ und $B(t)$ (what is more, a diffusion function of type $B(t)X_t$ has also been dismissed), but a linear model of the form

$$X_{t_o,t}=\Phi(t_o,t)X_{t_o}+c(t_o,t)+\tilde{Y}_{t_o,t}$$

is no doubt acçeptable and reasonable. In this model, $\tilde{Y}_{t_o,t}$ is a process which, strictly speaking for each t_o only, is stochastically independent of X_{t_o} and (according to data analysis) nearly Gaussian with a mean function 0. If, however, a workable model is required, we want an analytic form of the functions $\Phi(t_o,t)$ and $c(t_o,t)$, functions which so far have only been defined empirically, and we want this analytical form to be independent of any single stand. But this has not yet been achieved, for which reason we shall have to cope with the simplified model

$$X_t=\Phi(t_o,t)\cdot X_{t_o}+\tilde{Y}_{t_o,t}$$

which seems plausible on grounds of data analysis (fig. 2). This offers very good analytic models for

$$\Phi(t_o,t) \text{ and } \Psi^2(t_o,t)$$

(see (7) and (8), fig. 4a,b). In this model, too, $\tilde{Y}_{t_o,t}$ can be regarded as a centered, near-Gaussian process. However, the independence of X_{t_o} is no longer warranted, as we have forced the regression line through zero (fig. 2), with the result that according to the intercept sign we obtain a positive or negative correlation between residues and initial diameters at the age of t_o.

The equation which follows holds for the covariance function of the simplified process:

$$\text{Cov }(X_{t_o},X_t)=E(X_{t_o}-EX_{t_o})(X_t-EX_t)$$

$$=E(X_{t_o}-EX_{t_o})(\Phi(t_o,t)(X_{t_o}-EX_{t_o})+\tilde{Y}_{t_o})$$

$$=\Phi(t_o,t)\cdot\text{Var }X_{t_o}+\text{Cov}(X_{t_o},\tilde{Y}_{t_o,t})$$

Some thoughts concerning application as a prognosis model

The application of the model for the purpose of growth prognoses can be thought of as follows. A stand is observed from the age of t_o to s; thus we estimate individually the values of the parameters L, k, p, r, q per stand and carry out an adjustment of the diameter distribution by means of a theoretical distribution. The diameter distribution must then be calculated, with the help of the model, up to the age of t. As the covariances of X_{t_o} and $\tilde{Y}_{t_o,t}$ vary according to t and s, we are obliged to neglect them. According to our model

$$X_t = \hat{\Phi}(s,t) X_s + \tilde{Y}_{s,t}$$

$$\hat{\Phi}(s,t) = \frac{1-Le^{-kt}}{1-Le^{-ks}}$$

the diameter distribution at the age of t is derived from a convolution of the distribution of $\hat{\Phi}(s,t)X_s$ with a Gauss distribution which has the expected value zero and a variance of

$$\hat{\varphi}^2(s,t) = p(t-s)^r \cdot s^q$$

In the case of positive covariances, therefore, we must be particularly prepared for an underestimation of the variance of X_t. This error will be of importance in cases where $Cov(X_s, \tilde{Y}_{s,t})$ is high compared with Var X_s, which is why, as examples for our prognosis, we chose two cases from the Bueren stand ("schwache Niederdurchforstung"), where the correlation between residual process and initial distribution was extremely high. In both examples the model parameters were estimated from the overall growth development between the ages of 42 to 88, in order to avoid as best as possible any errors that may arise from changes in the general conditions of growth. It is in this sense only that the examples can be called unrealistic, but for the same reason they are more suitable for testing the model. In example 1, the initial distribution at the age of 71 was adjusted by a Pearson distribution of type I and computed, as described, up to the age of 85. Here the neglected correlation had the exceptionally high value of .63. Fig. 5 presents a comparison between the empirical distribution function at age 85 and the predicted one. Example 2 (correlation .44) is a prediction on the basis of age 63 for age 75 (fig. 6); the model parameters are the same and a Pearson Type I initial distribution was used. In example 1 we observe, in addition to a mean error caused by an adjustment error of function $\hat{\Phi}(s,t)$, some distinct deviations in the type of distribution and especially in the diameter variation, whilst in example 2 it is mainly the expected underestimation of the variance that gives us reason for concern. In valuating these model errors it should be kept in mind that the examples, with regard to the mutual dependence of X_s and $\tilde{Y}_{s,t}$, are utterly extreme cases taken from the considerable amount of material that we investigated.

The highest correlations observed are as follows:

stand	s	t	correlation
1	79	88	0.67
2	75	88	0.66
3	71	75	0.41
4	71	81	0.58
5	76	84	0.45
6	81	84	0.43
7	81	84	0.52
8	37	59	0.50

but as a rule one gets values that are considerably lower. For instance, in 57 out of 66 possible stand transitions (s,t) of Hauersteig 2, the correlations are below .3. Thus fig. 1 shows that model adjustment is generally much better. If one succeeds in incorporating into the model the temporal change of this correlation between initial distribution and residual process, it will of course also be possible to calculate exactly the distribution of X_t as the distribution of the sum of the dependent random variables X_{t_o} and $\tilde{Y}_{t_o,t}$.

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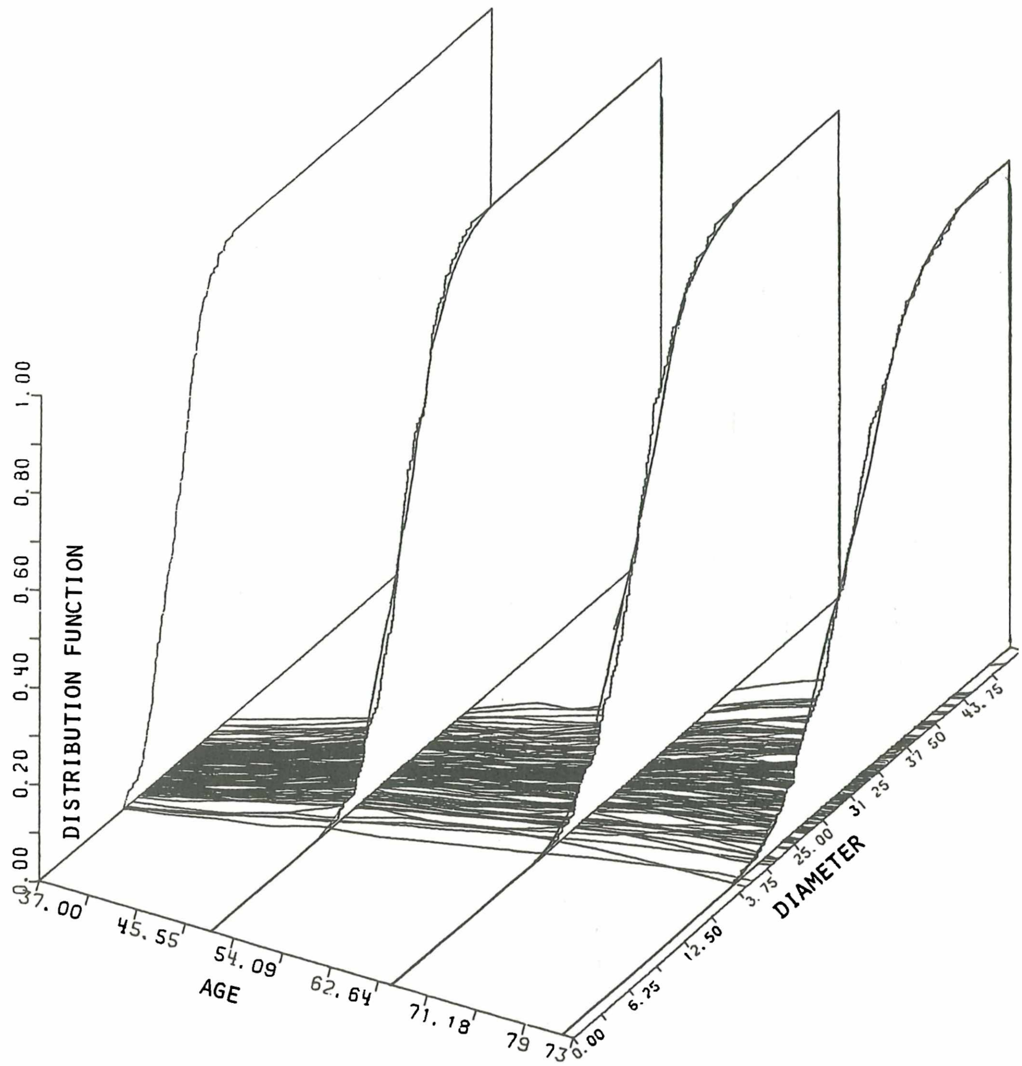
Fig. 1

Hauersteig 4 - Living Trees



Forecasting according to model (3')

Fig. 2

GRAPHICAL VERIFICATION OF MODEL $X_t=\Phi(t) X_{t_0} + c(t) + Y_t$ AS SOLUTION OF STOCH. DIFFERENTIAL EQUATION $dX_t=(A(t)X_t + a(t))dt + B(t)dW_t$

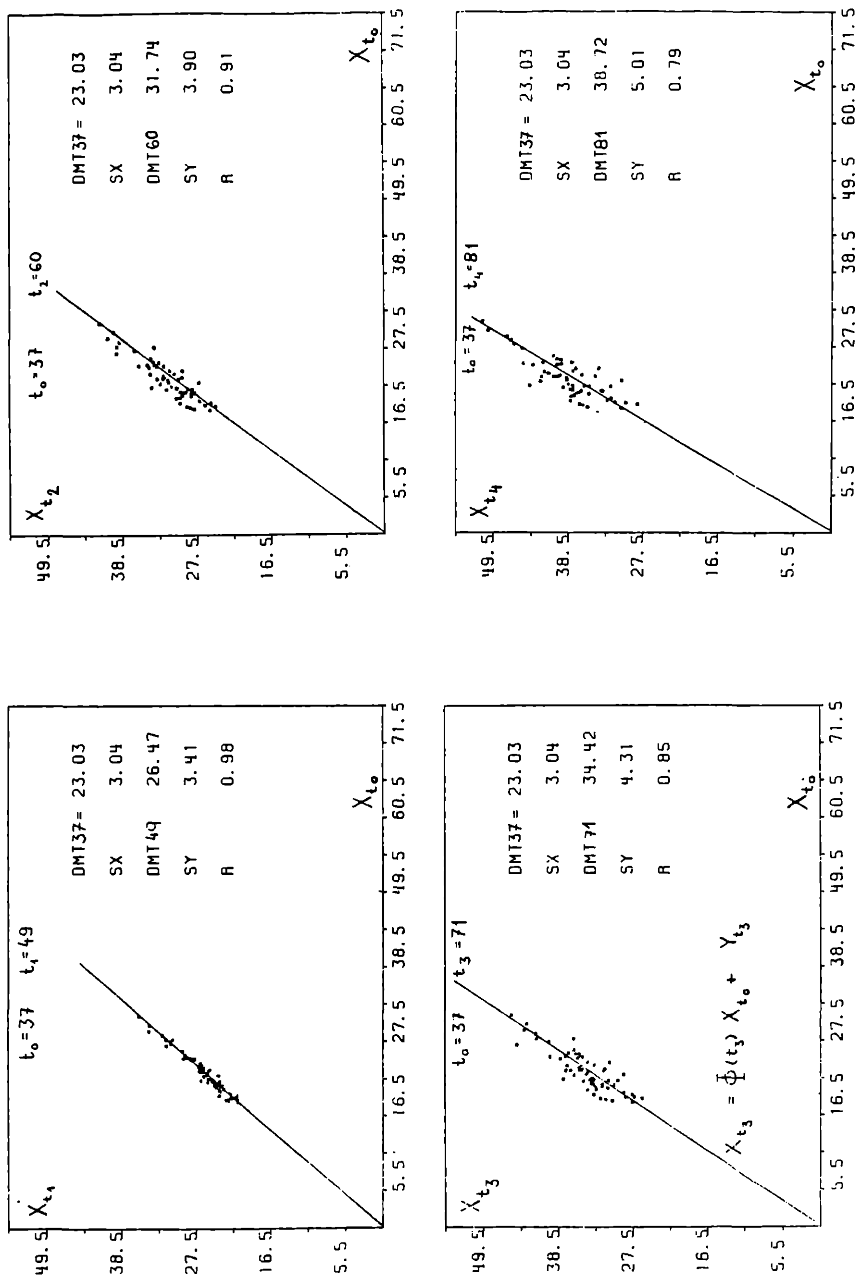


Fig.3

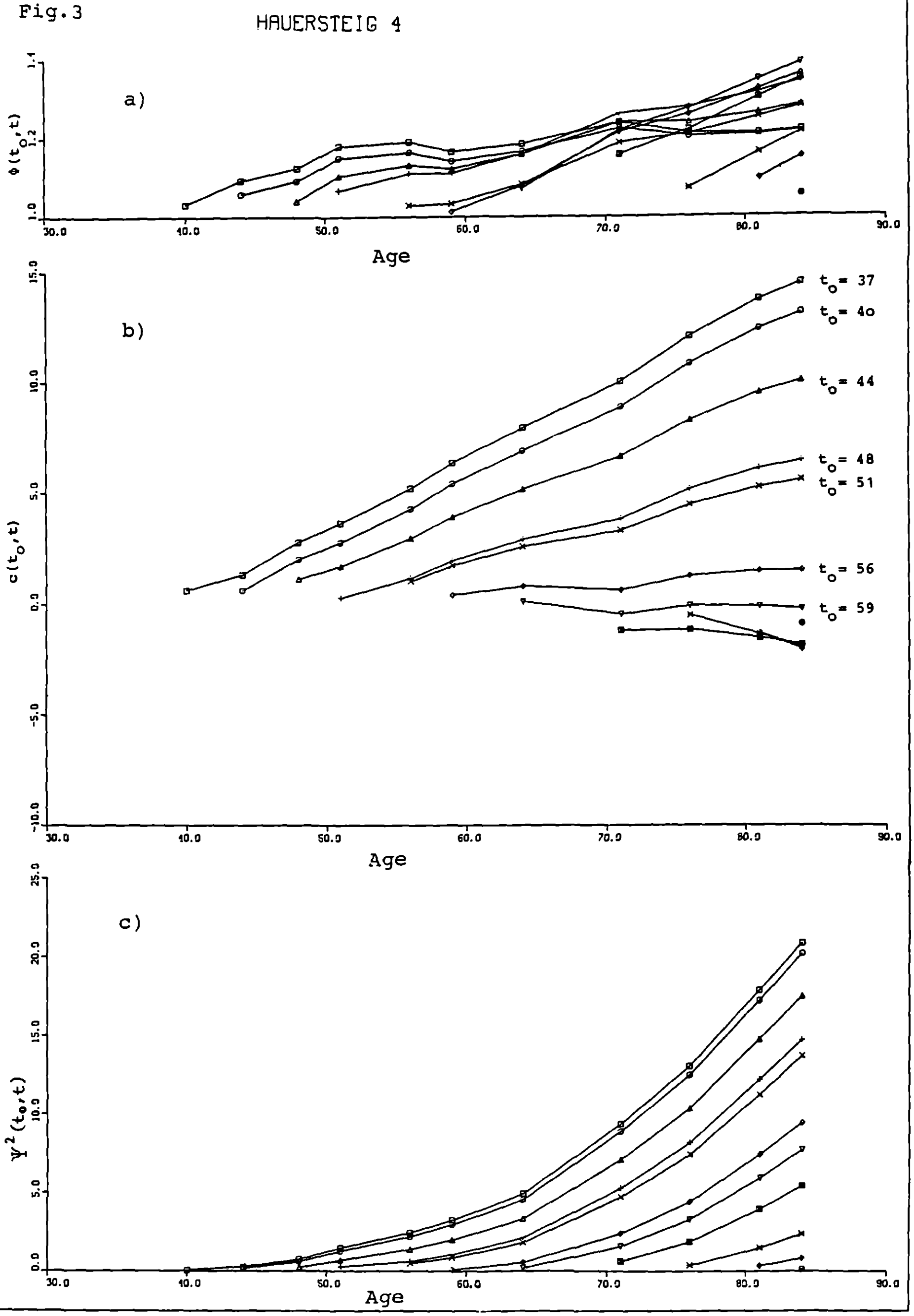

HAUERSTEIG 4
a)
φ(t_o,t)
1.0
1.2
1.4
30.0
40.0
50.0
60.0
70.0
80.0
90.0
Age
b)
c(t_o,t)
15.0
10.0
5.0
0.0
-5.0
-10.0
t_o= 37
t_o= 40
t_o= 44
t_o= 48
t_o= 51
t_o= 56
t_o= 59
30.0
40.0
50.0
60.0
70.0
80.0
90.0
Age
c)
Ψ²(t_o,t)
25.0
20.0
15.0
10.0
5.0
0.0
30.0
40.0
50.0
60.0
70.0
80.0
90.0
Age

Fig.4a)

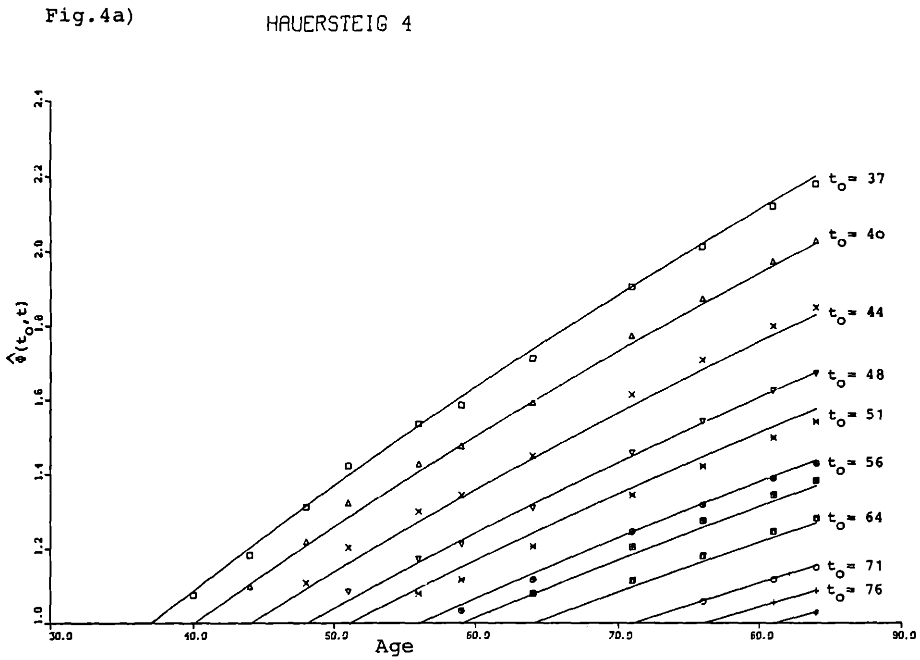

HAUERSTEIG 4
$\hat{\Phi}(t_o,t)$
Age
t_o= 37
t_o= 40
t_o= 44
t_o= 48
t_o= 51
t_o= 56
t_o= 64
t_o= 71
t_o= 76
30.0
40.0
50.0
60.0
70.0
80.0
90.0
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4

Fig.4b)

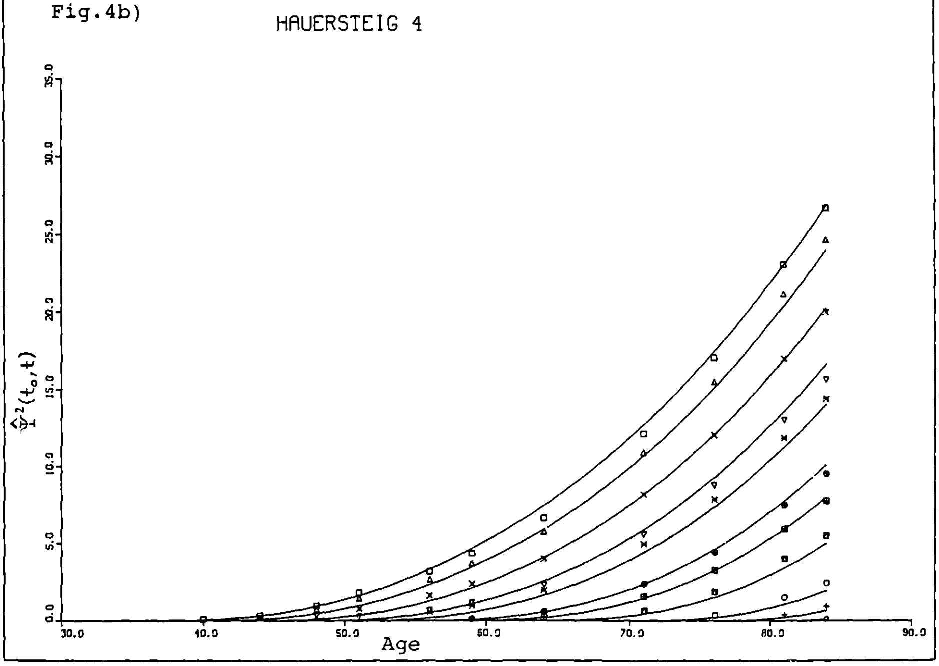

HAUERSTEIG 4
$\hat{\Psi}_1^2(t_o,t)$
Age
30.0
40.0
50.0
60.0
70.0
80.0
90.0
0.0
5.0
10.0
15.0
20.0
25.0
30.0
35.0

Fig.5

EXAMPLE 1

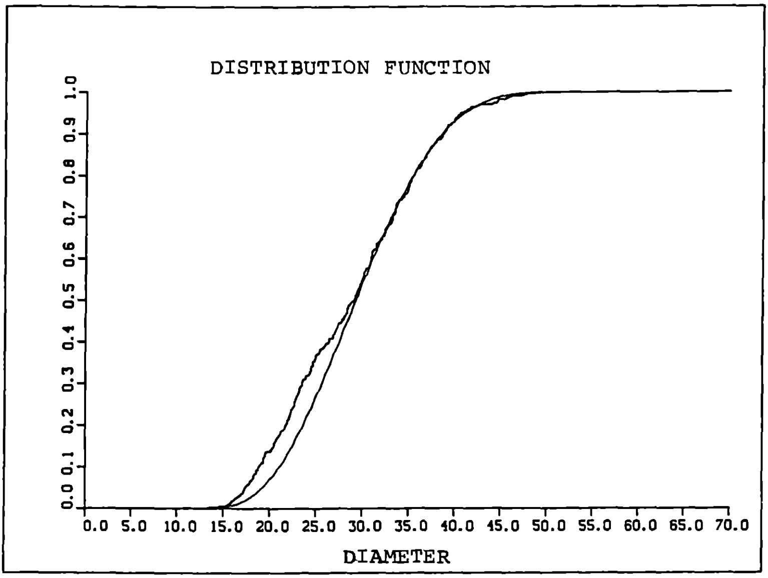


Fig.6

EXAMPLE 2