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On the Distribution of Formal Power Series Transformations with Respect to Embeddability in the Order Topology

By

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1. Introduction. The Main Results

Let $\mathbb{C}[[x_1, \ldots, x_n]]$ (briefly $\mathbb{C}\,[[x]]$, where $x = {}^t(x_1, \ldots, x_n)$ is the vector of indeterminates) be the ring of formal power series in n indeterminates $x_1, \ldots, x_n$ with complex coefficients. We consider in this paper formal power series transformations F by which we understand automorphisms F of $\mathbb{C}[[x]]$ which are continuous with respect to the order topology (i.e., order preserving) and leave every element of the ground field $\mathbb{C}$ fixed. It is well known that these automorphisms F are in $1-1$ correspondence to the images $F(x) = Ax + \mathcal{P}(x)$ of x. Here A runs through the matrices of $GL(n, \mathbb{C})$, and $\mathcal{P}(x)$ is an n-tuple of formal power series with $\mathrm{ord}(\mathcal{P}) \geq 2$. Moreover, these automorphisms form a group Γ under composition $\circ$ which is, in the above mentioned picture, represented by substitution of one n-tupel $Ax + \mathcal{P}(x) \in \mathbb{C}[[x]]$ into another.

F is called *iterable* (*embeddable*), if there exists a family $(F_t)_{t\in\mathbb{C}}$ in Γ such that

$$(T) \qquad F_t \circ F_s = F_{t+s},\ t, s \in \mathbb{C},$$

$$(E) \qquad F_1 = F.$$

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(T) is nothing but the famous translation equation, (E) is the embedding condition. $(F_t)_{t\in\mathbb{C}}$ is then clearly a group with operation $\circ$, called *iteration* (*group*) of F In general, such an iteration does not exist for a given F.

The problem of finding an embedding (iteration) of a selfmapping of a given set can be studied in much more general situations and is one of the main problems of iteration theory (see e.g. [19], [20]). The above mentioned iteration problem in Γ was studied in detail by several authors. We refer the reader to the survey articles [9], [10], [11], [16], as well as to [1], [2], [4]–[7], [18].

In a series of papers ([12], [14], [15]) we investigated how iterable (and also noniterable) power series tranformations are distributed locally in a neighbourhood of a given $F\in\Gamma$, where neighbourhood is understood in the sense of the *weak* topology (coefficientwise topology on $\mathbb{C}[[x]]$, resp. Γ). Now we are interested in the same type of problems, but the neighborhoods of the given $F\in\Gamma$ are those in the socalled *order* topology (*strong* topology).

The order topology on $\mathbb{C}[[x_1,\ldots,x_n]]$ is described by introducing the sets

$$U_N(\Phi) := \{\Psi\in\mathbb{C}[[x]]|\operatorname{ord}(\Phi-\Psi) > N\}$$

as the members of a basis of open neighborhoods of $\Phi\in\mathbb{C}[[x]]$, where N runs through $\mathbb{N}$. This leads then to the product topology on the space $(\mathbb{C}[[x]])^n$, and by identifying $F\in\Gamma$ with $F(x) = Ax + \mathcal{P}(x) \in (\mathbb{C}[[x]])^n$) and by restricting everything to Γ we get the order topology (strong topology) on Γ This topology can be introduced by a metric, and has therefore almost all good properties. The following basic results are easy to prove.

Lemma 1. *(i) The mapping from $\Gamma\times\Gamma$ to Γ, defined by $(F, G)\mapsto F\circ G$, is continuous. (ii) The mapping from Γ to Γ, defined by $F\mapsto F^{-1}$, is continuous.*

Proof: (i) If $(\nu_1, \ldots, \nu_n)\in\mathbb{N}_0^n$, let $|\nu| := \nu_1+\cdots+\nu_n$, and $x^\nu := x_1^{\nu_1}\cdots x_n^{\nu_n}$. We consider $F\in\Gamma$ as $F(x) = Ax + \mathcal{P}(x) \in (\mathbb{C}[[x]])^n$ Let $a_{k,\nu}$, $b_{k,\nu}$, $c_{k,\nu}$ be the coefficients of x^ν in the k-th components F_k, G_k $(F\circ G)_k$ $(1\le k\le n)$ of F, G, $F\circ G$, respectively. Then $c_{k,\nu}$ is a (universal) polynomial

$$c_{k,\nu} = \phi_{k,\nu}(a_{l,\mu}, b_{l,\mu}|1\le l\le n,\ 1\le|\mu|\le|\nu|)$$

in the coefficients $a_{l,\mu}$, $b_{l,\mu}$ of F and G, where $1\le l\le n$, $|\mu|\le|\nu|$. Hence, if for a certain $N\ge 1, \operatorname{ord}(F-\tilde{F}) > N$, $\operatorname{ord}(G-\tilde{G}) > N$, then we have for the corresponding coefficients $\tilde{a}_{l,\mu}, \tilde{b}_{l,\mu}$ of $\tilde{F}$, $\tilde{G}$

$$\tilde{a}_{l,\mu} = a_{l,\mu},\ \tilde{b}_{l,\mu} = b_{l,\mu},\quad l = 1,\ldots,n, |\mu|\le N,$$

and therefore

$$c_{k,\nu} = \phi_{k,\nu}(a_{l,\mu}, b_{l,\mu} | 1 \le l \le n,\ 1 \le |\mu| \le |\nu|) = \\ = \phi_{k,\nu}(\tilde{a}_{l,\mu}, \tilde{b}_{l,\mu} | 1 \le l \le n,\ 1 \le |\mu| \le |\nu|) = \tilde{c}_{k,\nu}$$

for $k = 1, \ldots, n$, provided that $|\nu| \le N$. From this we deduce $\operatorname{ord}(F \circ G - \tilde{F} \circ \tilde{G}) > N$, establishing Lemma 1(i).
(ii)F^{-1} is represented by an n-tuple of the form $F^{-1} = A^{-1}x + \mathcal{Q}(x)$, if $F(x) = Ax + \mathcal{P}(x)$. The coefficient $d_{k,\nu}$ of x^ν in $\mathcal{Q}_k(x)$ is represented by a (universal) function

$$d_{k,\nu} = \psi_{k,\nu}(A, a_{l,\mu} | 1 \le l \le n, 2 \le |\mu| \le |\nu|),$$

which is a rational function in the elements of $A \in GL(n, \mathbb{C})$ and a polynomial in $a_{l,\mu}$. Hence, if $\operatorname{ord}(F - \tilde{F} > N,\ (N \ge 1)$, then, denoting by $\tilde{a}_{k,\nu}$, $\tilde{d}_{k,\nu}$ the coefficients of $\tilde{F}$ and $\tilde{F}^{-1}$, respectively, we find

$$d_{k,\nu} = \psi_{k,\nu}(A, a_{l,\mu} | 1 \le l \le n, 2 \le |\mu| \le |\nu|) = \\ = \psi_{k,\nu}(A, \tilde{a}_{l,\mu} | 1 \le l \le n, 2 \le |\mu| \le |\nu|) = \tilde{d}_{k,\mu},$$

for $k = 1, \ldots, n$ and $|\nu| \le N$. Hence $F \mapsto F^{-1}$ is continuous in the order topology. ■

Lemma 2. *(i) $(\Gamma, \circ)$ is a topological group in the strong topology. (ii) For each $T \in \Gamma$ the conjugation c_T: $\Gamma \to \Gamma$, $c_T(F) := T^{-1} \circ F \circ T$ is an isomorphism of the topological group Γ.*

Proof: Immediate consequence of Lemma 1.

As can be seen from the survey articles [9], [10], [11], [16], a basic tool in solving the iteration problem in Γ are the socalled *semicanonical forms* (briefly *normal forms*) of the elements of Γ under conjugation. It is almost obvious that iteration problems and their possible solutions are invariant under conjugation. So we always may replace, for our purposes, a given F by one of its semicanonical forms (or a set of formal power series transformations by their simultaneous conjugates under the same c_T). We refer the reader to the papers quoted above for details about normal forms and their applications in solving the iteration problem.

We are now ready to sketch the type of problems we will deal with. Let $F \in \Gamma$. Under what conditions on F does there exist a neighbourhood $U_N(F)$ of F (in the strong topology) consisting entirely of iterable (or entirely of noniterable) power series transformations? Under what conditions on a given F is there in each neighbourhood of F an iterable $G \ne F$, or a noniterable $G \ne F$, or are there in each neighbourhood both iterable

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and noniterable power series transformations? Is the limit of a convergent series of iterable automorphisms always iterable?

In Section 2 we will discuss series in one indeterminate, and give a rather complete answer. We can do this, since the semicanonical forms in this case are well known. There are even strict trinomial normal forms due to Scheinber ([7]) which we will also use in one place.

The main difference between the weak and the strong topology – as far as our investigations are concerned-, seems to lie in the fact that there are power series $F(z) = \rho z + c_2 z^2 + \cdots$ (they are indeed the nonembeddable ones) with the property that they have strong neighborhoods which consist of noniterable series. In the weak topology one can prove (independently of the number of indeterminates) that in each (weak) neighbourhood of any given $F \in \Gamma$ there are iterable automorphisms G, different from F.

In Section 2 we will also show that in the 1-dimensional case the limit of each convergent sequence of iterable series is iterable. The problem remains open in higher dimensions. In the weak topology a convergent sequence of iterable series need not have an iterable limit, not even in dimension 1. These results concerning the distribution with respect to the weak topology are proved in the papers [12], [14] and [15] quoted above.

Our techniques yield some insight to a more detailed structure of convergent sequences (in the 1-dimensional case). Roughly speaking, if the limit F of a convergent sequence $(F_l)_{l \in \mathbb{N}}$ of formal power series is of the form $F(z) = \rho z + c_2 z^2 + \cdots$ where a) ρ is not a root of 1, or b) $F(z)$ is not iterable, or c) $F(z) = z + c_2 z^2 + \cdots$, but $F(z) \neq z$, then almost all F_l's ($l \geq l_0$) are conjugate to F. Also, if almost all F_l's are iterable and ρ in the limit $F(z) = \rho z + c_2 z^2 + \cdots$ is different from 1, the same is true.

In the higher-dimensional case of the automorphism group Γ of $\mathbb{C}[\![x_1, \ldots, x_n]\!]$ $(n > 1)$ our results are far from complete. Let $F \in \Gamma$, $F(x) = Ax + \mathcal{P}(x)$, where $A \in GL(n, \mathbb{C})$, $\mathrm{ord}(\mathcal{P}) \geq 2$, and where $\rho_1, \ldots, \rho_n$ are the eigenvalues of A. Denote by $\mathfrak{R}$ the set of relations $\rho_k = \rho_1^{\nu_1} \cdots \rho_n^{\nu_n}$, $1 \leq k \leq n$, with $\nu_j \in \mathbb{N}_0$, $\nu_1 + \cdots + \nu_n \geq 2$. If $\mathfrak{R}$ is finite (possibly empty) then we can show that for an iterable [a noniterable] F there exists a strong neighbourhood $U_N(F)$ which contains only iterable [noniterable] automorhphisms. If $\mathfrak{R}$ is finite, then a convergent sequence $(F_{(l)})_{l \in \mathbb{N}}$ of iterable [noniterable] $F_{(l)}$'s has an iterable [noniterable] limit.

If no assumptions on the eigenvalues are imposed then we can only prove that if $F \in \Gamma$ is iterable [noniterable], then there is a sequence $(F_{(l)})_{l \in \mathbb{N}}$ of iterable [noniterable] power series transformations converging to F and such that $F \neq F_{(l)}$ for all l. If $F \in \Gamma$ has a neighbourhood

$U(F)$ such that each $G \in U(F)\setminus\{F\}$ is iterable [noniterable], then F is iterable [noniterable]. In this result we are able to replace the exceptional set $\{F\}$ by a somewhat larger set, but unfortunately up to now not by a 'big' set. Eventually, we will prove a result on semicanonical forms $N_{(l)}$ of the members $F_{(l)}$ of a convergent sequence $(F_{(l)})_{l\in\mathbb{N}}$ in Γ, but weaker than similar results in Section 2 for the case $n=1$. Namely, there is $l_0 \in \mathbb{N}$ such that for $l \geq l_0$ each $F_{(l)}$ can be represented as $F_{(l)} = V_{(l)}^{-1} \circ N_{(l)} \circ V_{(l)}$, where $N_{(l)}$ is a semicanonical form of $F_{(l)}$, $N = \lim_{l\to\infty} N_{(l)}$ and $V = \lim_{l\to\infty} V_{(l)}$ exist and $\lim_{l\to\infty} F_{(l)} = V^{-1} \circ N \circ V$ with the semicanonical form N.

2. Power Series Transformations in One Indeterminate

2.1. Problems of Distribution

We recall (see [12]) some basic results on semicanonical forms of power series in one indeterminate and on the connection of iterability and normal forms. By E we denote the group of complex roots of 1.

Lemma 3. *(i) If $F(z) = \rho z + c_2 z^2 + \cdots$, $\rho \in \mathbb{C}^{\cdot} \setminus E$, then F is conjugate to its linear part ρz. (ii) If $F(z) = \rho z + c_2 z^2 + \cdots$, $\rho \in E \setminus \{1\}$, and if $\rho = \exp(2\pi i\alpha/\beta)$ with $\alpha, \beta \in \mathbb{Z}, \beta > 1$, $\gcd(\alpha, \beta) = 1$, then F is conjugate to a semicanonical form*

$$N(z) = \rho z + \sum_{\nu\geq 1} \varphi_\nu z^{\nu\beta+1}$$

which is, in general, not uniquely determined. If ρz is a semicanonical form of F, then ρz is the only one. If, on the other hand, F has a semicanonical form $N(z) = \rho z + \varphi_{\nu_0} z^{\nu_0\beta+1} + \cdots$, where $\nu_0 \geq 1$, $\varphi_{\nu_0\beta+1} \neq 0$, then for each semicanonical form $M(z)$ of F the series $M(z) - \rho z$ has order $\nu_0\beta + 1$. (iii) If $F(z) = z + d_k z^k + \cdots$ where $k \geq 2$, $d_k \neq 0$, then each conjugate $T^{-1} \circ F \circ T$ of F has the same form $(T^{-1} \circ F \circ T)(z) = z + e_k z^k + \cdots$, with $e_k \neq 0$. ■

Lemma 4. *(i) Assume that $\rho \neq 1$ and $F(z) = \rho z + c_2 z^2 + \cdots$. Then F is iterable if and only if it is conjugate to its linear part ρz. Hence, if $\rho \in \mathbb{C}^{\cdot}\setminus E$, then F is always iterable. If $\rho \in E \setminus \{1\}$, then F is iterable iff each semicanonical form is linear (iff at least one semicanonical form is linear). (ii) If $F(z) = z + d_k z^k + \cdots$ with $k \geq 2$, $d_k \neq 0$, then F is iterable.* ■

From these lemmas we deduce a survey on the local distribution of formal series with respect to embeddability in the strong topology. Theorems 1 and 2 are already contained in [12], but for reasons of completeness we reproduce them here. For Theorem 2 we will give a new, farther reaching, proof too.

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Theorem 1. *(i) If $F(z) = \rho z + \Sigma_{\nu \geq 2} c_\nu z^\nu$, where $\rho \in \mathbb{C}^{\cdot} \setminus E$, then each sufficiently small neighbourhood of F (in the order topology) contains only iterable series. (ii) If $F(z) = z + \cdot$, then each sufficiently small neighbourhood of F consists entirely of iterable series. (iii) If $F(z) = \rho z + \cdot$, with $\rho \in E \setminus \{1\}$, is iterable, then in each neighborhood of F there are iterable power series, different from F, as well as noniterable series. (iv) If $F(z) = \rho z + \cdot$, with $\rho \in E \setminus \{1\}$, is noniterable, then each sufficiently small neighbourhood of F consists entirely of noniterable series.*

Proof: (i) Take the neighbourhood $U_1(F)$. Then each $G \in U_1(F)$ starts with ρz, where $\rho \in \mathbb{C}^{\cdot} \setminus E$. Hence, according to Lemma 4(i), G is iterable.
(ii) Here each G in $U_1(F)$ has the form $z + \cdot$, so is either the identity (hence iterable) or is iterable by Lemma 4 (ii).
(iii) Suppose now that $F(z) = \rho z + c_2 z^2$ where $\rho \in E \setminus \{1\}$, is iterable. ρ can be represented as $\rho = \exp(2\pi i \alpha / \beta)$, with $\alpha, \beta \in \mathbb{Z}$, $\beta > 1$, $\gcd(\alpha, \beta) = 1$. According to Lemma 4(i) there is a $T \in \Gamma$, $T(z) = z + t_2 z^2 + \cdot$ such that $(T^{-1} \circ F \circ T)(z) = \rho z$. As we know (see Section 1), we may assume for our purposes that $F(z) = \rho z$. Consider first the sequence $(F_n)_{n \in \mathbb{N}}$ with $F_n(z) = \rho z + z^{n\beta + 1}$ for $n \in \mathbb{N}$. According to Lemmas 3 and 4, these F_n are in semicanonical form and are not linear and also not linearizable. Then, by Lemma 4, they are not iterable. Moreover, we have $\lim_{n \to \infty} F_n(z) = F(z) = \rho z$. This shows that each neighbourhood of F contains a series which is not iterable.

Consider now a sequence $(S_n)_{n \in \mathbb{N}}$ in Γ, where $S_n(z) = z + z^{k_n}$, and $(k_n)_{n \in \mathbb{N}}$ is strictly increasing and, furthermore, z^{k_n} is not an additional monomial with respect to the relation $\rho^{\beta+1} = \rho$, i.e. $k_n \not\equiv 1 \pmod{\beta}$, for each n. If we calculate the series $G_n := S_n^{-1} \circ F \circ S_n$ for the normal form $F(z) = \rho z$, we find

$$G_n(z) = \rho z + (\rho^{k_n} - \rho) z^{k_n} +$$

where $\rho^{k_n} - \rho \neq 0$. Hence $G_n \neq F$ for each n, $\lim_{n \to \infty} G_n(z) = F(z)$, and $G_n = S_n^{-1} \circ F \circ S_n$ is iterable since it is conjugate to F. Therefore we have constructed an iterable power series G, different from F, in each neighbourhood of F.
(iv) The last case refers to a series $F(z) = \rho z + c_2 z^2 + \cdot \cdot$, where $\rho \in E \setminus \{1\}$, and F is not iterable. Again, we may assume that F is already a semicanonical form. So, if $\rho = \exp(2\pi i \alpha / \beta)$, α, $\beta \in \mathbb{Z}$, $\beta > 1$, $\gcd(\alpha, \beta) = 1$, then by Lemmas 3 and 4

$$F(z) = \rho z + d_{\nu_0 \beta + 1} z^{\nu_0 \beta + 1} + \sum_{\nu > \nu_0} d_{\nu\beta+1} z^{\beta+1},$$

where $\nu_0 \geq 1$ and $d_{\nu_0\beta+1} \neq 0$. Then each series $G \in U_{\nu_0\beta+1}(F)$ is of the form

$$G(z) = \rho z + d_{\nu_0\beta+1} z^{\nu_0\beta+1} + \sum_{\mu>\nu_0\beta+1} g_\mu z^\mu$$

If we want to construct a semicanonical form of G, then, as is well known from the theory of normal forms, we can achieve this through conjugation by a transformation

$$V(z) = z + \sum_{\mu>\nu_0\beta+1} v_\mu z^\mu,$$

since the $(\nu_0\,\beta+1)$-jet of G is already in semicanonical form with respect to ρ ('Formales Ausfegen', cf. [3], ch. 3). This means that

$$(V^{-1} \circ G \circ V)(z) = \rho z + d_{\nu_0\beta+1} z^{\nu_0\beta+1} + \sum_{\nu>\nu_0} \tilde{d}_{\nu\beta+1} z^{\nu\beta+1},$$

and since $d_{\nu_0\beta+1} \neq 0$, $V^{-1} \circ G \circ V$ and G are not iterable, according to Lemma 4. ■

Similar techniques allow us to show that in the case of one indeterminate each convergent sequence of iterable power series has an iterable limit (Theorem 1 shows that a sequence of noniterable series may have an iterable limit).

Theorem 2. *Let $(F_l)_{l\in\mathbb{N}}$ be a convergent sequence of iterable power series transformations in one indeterminate, and let $F = \lim_{n\to\infty} F_l$. Then F is iterable.*

Proof: We assume that the limit F of $(F_l)_{l\in\mathbb{N}}$ is $F(z) = \rho z + c_2 z^2 + \cdots$ If $\rho \in \mathbb{C}^{\cdot} \setminus E$ or $\rho = 1$, then F is iterable by Lemma 4. So it suffices to consider the case $\rho \in E \setminus \{1\}$. Assume $\rho = \exp(2\pi i\alpha/\beta)$, α, $\beta \in \mathbb{Z}$, $\beta > 1$, $\gcd(\alpha, \beta) = 1$. Then there is an index l_0 such that for $l \geq l_0$ $F_l(z) = \rho z + c_2^{(l)} z^2 + \cdots$, and each F_l is iterable by assumption. We assume now that F is not iterable. Take any normal form of F:

$$(T^{-1} \circ F \circ T)(z) = \rho z + \sum_{\nu\geq 1} d_{\nu\beta+1} z^{\nu\beta+1}$$

From Lemma 4 we know that there is an index $\nu_0 \geq 1$ such that

$$(T^{-1} \circ F \circ T)(z) = \rho z + d_{\nu_0\beta+1} z^{\nu_0\beta+1} + \sum_{\nu\geq\nu_0} d_{\nu\beta+1} z^{\nu\beta+1},$$

where $d_{\nu_0\beta+1} \neq 0$. Instead of F and $(F_l)_{l\in\mathbb{N}}$ we may consider $T^{-1} \circ F \circ T$ and the sequence $(T^{-1} \circ F_l \circ T)_{l\in\mathbb{N}}$ of iterable transformations. Now, there

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is an $l_1 \geq l_0$ such that for each $l \geq l_1$

$$(T^{-1} \circ F_l \circ T)(z) = \rho z + d_{\nu_0\beta+1} z^{\nu_0\beta+1} + \sum_{\mu>\nu_0\beta+1} \tilde{d}_\mu z^\mu,$$

with $d_{\nu_0\beta+1} \neq 0$. We have already shown in the proof of Theorem 1 that F_l is not iterable which is a contradiction to the assumption. Hence F must be iterable. ■

There is a different proof of Theorem 2 which, moreover, gives more information on convergent sequences of iterable power series. We will present it here. We again consider the case $\rho \in E\backslash\{1\}$, $\rho = \exp(2\pi i\alpha/\beta)$, and use the same notation.

Theorem 2, alternative proof. Since each F_l is iterable, it is of the form $F_l(z) = T_l^{-1}(\rho T_l(z))$ for $l \geq l_0$, where $T_l(z) = z + t_2^{(l)} z^2 + \cdots$ The series $(T_l)_{l\in\mathbb{N}}$ is, in general, not uniquely determined by F_l. However, there is for each F_l a unique normalized T_l, namely normalized by the condition

$$T_l(z) = z + \sum_{\substack{\mu>1 \\ \mu\not\equiv 1 (\mathrm{mod}\ \beta)}} t_\mu^{(l)} z^\mu,$$

i.e., by $t_\mu = 0$ if $\mu \equiv 1$ (mod β) (see [13]). The coefficients $t_\mu^{(l)}$ are universal functions

$$t_\mu^{(l)} = \phi_\mu(\rho; c_2^{(l)}, \ldots, c_\mu^{(l)}),$$

rational in ρ and polynomial in the coefficients $c_\nu^{(l)}$ of F_l. We see that $c_2^{(l)}, \ldots, c_m^{(l)}$ are the same as $c_2, \ldots, c_m$ for $l \geq L(m)$, hence $\phi_\mu(\rho; c_2^{(l)}, \ldots, c_\mu^{(l)}) = \phi_\mu(\rho; c_2, \ldots, c_\mu)$ for $\mu \leq m$ and $l \geq L(m)$. This means that $\lim_{l\to\infty} T_l = T$ exists, and $T(z) = z + \sum_{\nu\geq 2} t_\nu z^\nu$. Hence (see Section 1, Lemma 1)

$$F(z) = \lim_{l\to\infty} T_l^{-1}(\rho T_l(z)) = T^{-1}(\rho T(z)),$$

and F is iterable. ■

This proof shows in addition that, in the case under consideration, $F_l = T_l^{-1}(\rho T_l(z))$ for $l \geq l_0$, where $\lim_{l\to\infty} T_l = T$ exists and $F(z) = \lim_{l\to\infty} T_l^{-1}(\rho T_l(z)) = T^{-1}(\rho T(z))$.

Similar results hold in other cases. As a first result we present.

Theorem 3. *Let $(F_l)_{l\in\mathbb{N}}$ be a convergent sequence of iterable series F_l, let $F = \lim_{l\to\infty}$ and assume that $F(z) = \rho z + \cdots$, where $\rho \neq 1$. Then, for sufficiently large l ($l \geq l_0$),* *there exists a sequence* $(T_l)_{l\geq l_0}$, $T_l(z) = z + \cdots$, *such that*

a) $\lim_{l\to\infty} T_l = T$ $(T(z) = z + \cdots)$ *exists,*
b) $F_l(z) = T_l^{-1}(\rho T_l(z))$,
c) $F(z) = T^{-1}(\rho T(z))$.

Proof: We already gave the proof for the case where $\rho\in E\setminus\{1\}$. Assume now that $\rho\in\mathbb{C}^{\cdot}\setminus E$. It is obvious that for $l\geq l_0$ $F_l(z) = \rho z + c_2^{(l)}z^2 + \cdots$ According to Lemma 3 there are transformations V_l, $V_l(z) = z + \sum_{\mu\geq 2} v_\mu^{(l)} z^\mu$, such that $F_l(z) = V_l^{-1}(\rho V_l(z))$. The V_l's are unique, in fact, their coefficients $v_\mu^{(l)}$ are universal functions $v_\mu^{(l)} = \psi_\mu(\rho; c_2^{(l)}, \ldots, c_\mu^{(l)})$, rational in ρ, polynomial in $c_2^{(l)}, \ldots, c_\mu^{(l)}$. To each $m\geq 1$ there is an $L(m)$ such that $c_\mu^{(l)} = c_\mu$, for $2\leq\mu\leq m$ and $l\geq L(m)$. Hence $v_\mu^{(l)} = \psi_\mu(\rho; c_2^{(l)}, \ldots, c_\mu^{(l)}) = \psi_\mu(\rho; c_2, \ldots, c_\mu)$ for $\mu = 2, \ldots, m$, if $l\geq L(m)$. This means again that $\lim_{l\to\infty} V_l = V$ exists and $F = \lim_{l\to\infty} F_l = V^{-1}(\rho V(z))$. ■

2.2. Convergent Sequences

Theorem 3 leads to the question whether in the case of power series transformations in one indeterminate more details can be derived about the structure of sequences $(F_l)_{l\in\mathbb{N}}$ in Γ which converge in the order topology. Let F be the limit of $(F_l)_{l\in\mathbb{N}}$. If, e.g., $F(z) = \rho z$, where $\rho\in E\setminus\{1\}$ (more generally, if F is iterable and has a multiplier $\rho\in E\setminus\{1\}$), then we have already seen that F is the limit of a sequence of iterable series G_l, $G_l\neq F$, and also the limit of a sequence of noniterable ones. Hence in this case we cannot expect to find more details about the structure of sequences converging to F. The same happens if $F(z) = z$. But if $\rho\in E\setminus\{1\}$, $F(z) = \rho z + \cdots$, and F is noniterable, or in the case where $F(z) = z + d_k z^k + \cdots$ with $d_k\neq 0$ for some $k\geq 2$, we will show a result about the sequences $(F_l)_{l\in\mathbb{N}}$, converging to F, which is very similar to Theorem 3. For this purpose we need the trinomial normal forms of Scheinberg ([17]).

Theorem 4. *Let* $(F_l)_{l\in\mathbb{N}}$ *be a convergent sequence*, $\lim_{l\to\infty} F_l = F$, and $F(z) = z + d_n z^n + \cdots$, *where* $n\geq 2$, $d_n\neq 0$. *Let* $N(z) = z + az^n + bz^{2n-1}$ *be the trinomial normal form of* F, $N = S^{-1}\circ F\circ S$, *with some* $S(z) = z + \cdots \in\Gamma$ (*which implies* $a = d_n\neq 0$). *Then there exists an index* $l_0\in\mathbb{N}$ *and a sequence* $(T_l)_{l\geq l_0}\in\Gamma$, $T_l(z) = z + \cdots$, *and a solution* T, $T(z) = z + \cdots$, *of* $F = T^{-1}\circ N\circ T$ *such that* (*i*)$\lim_{l\to\infty} T_l = T$, *and* (*ii*) $F_l = T_l^{-1}\circ N\circ T_l$, *for all* $l\geq l_0$.

In particular, for $l\geq l_0$, *all* F_l*'s are conjugate to* F.

Proof: There is an l_0 such that, for $l\geq l_0$, $F_l(z) = z + d_n z^n + d_{2n-1}z^{2n-1} + \sum_{\mu\geq 2n} d_\mu^{(l)} z^\mu$, if $F(z) = z + d_n z^n + d_{2n-1}z^{2n-1} + \sum_{\mu\geq 2n} d_\mu z^\mu$ For F and

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each F_l, $l \geq l_0$, there exist transformations T, $T(z) = z + \cdot$ and $T_l, T_l(z) = z + \quad \cdot$, such that

$$(T^{-1} \circ F \circ T)(z) = (T_l^{-1} \circ F_l \circ T_l) = z + az^n + bz^{2n-1},$$

with $a = d_n$. A reformulation of the proof of Proposition 6 in [17] tells us, how specific solutions of these Schröder type equations $T^{-1} \circ F \circ T = N$, $T_l^{-1} \circ F_l \circ T_l = N$ can be constructed. In fact, this T has the following structure:

$$T(z) = z + t_2 z^2 + \quad + t_{n-1} z^{n-1} + \sum_{\nu \geq 1} t_{n+\nu} z^{n+\nu}$$

where t_μ is a certain polynomial

$$t_\mu = p_\mu(d_n, d_{n+1}, . \quad , d_{n+\mu-1})$$

for $2 \leq \mu \leq n-1$, while, for $\nu \geq 1$, $t_{n+\nu}$ is a certain polynomial

$$t_{n+\nu} = p_{n+\nu}(d_n, d_{n+1}, \ldots, d_{2n-1}; d_{2n}, . \quad , d_{2n+\nu-1}).$$

(t_n may be chosen as 0, which we do.) Similarly,

$$T_l(z) = z + t_2 z^2 + \quad + t_{n-1} z^{n-1} + \sum_{\nu \geq 1} t^{(l)}_{n+\nu} z^{n+\nu}$$

where

$$t^{(l)}_\mu = t_\mu = p_\mu(d_n, d_{n+1}, .. \quad , d_{n+\mu-1})$$

for $2 \leq \mu \leq n-1$, $t^{(l)}_n = 0$, and for $\nu \geq 1$

$$t^{(l)}_{n+\nu} = p_{n+\nu}(d_n, d_{n+1}, \ldots, d_{2n-1}; d^{(l)}_{2n}, . \quad , d^{(l)}_{2n+\nu-1}).$$

Since $\lim_{l \to \infty} F_l = F$, we deduce $\lim_{l \to \infty} T_l = T$. Moreover, for $l \geq l_0$, F_l and F have the same trinomial normal form. ■

An analogue of Theorem 4 holds true in the case of convergent sequences $(F_l)_{l \in \mathbb{N}}$ whose limit is not iterable. Hence F is of the form $\rho z + c_2 z^2 + \cdot$ where $\rho \in E \setminus \{1\}$, say $\rho = \exp(2\pi i \alpha/\beta)$, α, $\beta \in \mathbb{Z}$, $\beta > 1$, $\gcd(\alpha, \beta) = 1$, and F is not linearizable.

Theorem 5. *Let $(F_l)_{l \in \mathbb{N}} \subset \Gamma$ be a convergent sequence with noniterable limit F. Then there is an l_0 and a sequence $(V_l)_{l \geq l_0}$ such that*

(i) $\lim_{l \to \infty} V_l = V$ exists, $V_l(z) = z + \cdot\cdot \quad , V(z) = z + \cdot \quad$, and
(ii) $F = V^{-1} \circ N \circ V$, $F_l = V_l^{-1} \circ N \circ V_l$ for $l \geq l_0$,

where N is the Scheinberg trinomial normal form of F.

Proof: There is an l_0 such that for all $l \geq l_0$ F_l is not iterable. Otherwise we would have a subsequence of iterable series of $(F_l)_{l \in \mathbb{N}}$, converging to F,

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and Theorem 2 would imply that F is also iterable. If l_0 is sufficiently large, then for $l \geq l_0$

$$F(z) = \rho z + c_2 z^2 + \quad \text{and}$$
$$F_l(z) = \rho z + c_2^{(l)} z^2 +$$

where $\rho = \exp(2\pi i \alpha/\beta)$, $\alpha, \beta \in \mathbb{Z}$, $\beta > 1$, $\gcd(\alpha, \beta) = 1$. We consider now the sequence $(F_l^\beta)_{l \in \mathbb{N}}$. From Section 1 it follows that $\lim_{l\to\infty} F_l^\beta = F^\beta$ Furthermore $F^\beta = z + \cdots$, $F_l^\beta(z) = z +$ for $l \geq l_0$. If l_0 is large enough, then we also have

$$F^\beta(z) = z + d_n z^n + \quad + d_{2n-1} z^{2n-1} + \sum_{\mu \geq 2n} d_\mu z^\mu,$$
$$F_l^\beta(z) = z + d_n z^n + \quad + d_{2n-1} z^{2n-1} + \sum_{\mu \geq 2n} d_\mu^{(l)} z^\mu,$$

where $n > 2$, $n-1$ is a multiple of β, and $d_n \neq 0$. Theorem 4 gives us the existence of transformations T and T_l, $l \geq l_0$, such that $F^\beta = T^{-1} \circ N_0 \circ T$, $F_l^\beta = T_l^{-1} \circ N_0 \circ T_l$, $l \geq l_0$, and $\lim_{l\to\infty} T_l = T$, where N_0 may be taken as the Scheinberg trinomial normal form of F^β,

$$N_0(z) = z + a z^n + b z^{2n-1},\ a \neq 0.$$

Here we apply Theorem 9 in [17] which states that also

$$F = T^{-1} \circ N_1 \circ T,\ \ F_l = T_l^{-1} \circ N_1 \circ T_l$$

for $l \geq l_0$, N_1 being an appropriate iterative root of order β of N_0, with multiplier ρ. If N is the trinomial normal form of N_1 (cf. [17], Prop. 10), then $S^{-1} \circ N \circ S = N_1$ and consequently $F = (S \circ T)^{-1} \circ N \circ (S \circ T)$, and $F_l = (S \circ T_l)^{-1} \circ N \circ (S \circ T_l)$ for $l \geq l_0$. Putting $V := S \circ T$, $V_l := S \circ T_l$, we also have $\lim_{l\to\infty} V_l = V$ and $F = V^{-1} \circ N \circ V$, $F_l = V_l^{-1} \circ N \circ V_l$ for $l \geq l_0$, which proves Theorem 5. ■

Corollary. *Let* $(F_l)_{l \in \mathbb{N}}$ *be a convergent sequence*, $\lim_{l\to\infty} F_l = F$, $F(z) = \rho z + c_2 z^2 +$ *Let us make one of the following assumptions*:

a) $\rho \notin E$, *or*

b) $F(z) = z + c_2 z^2 +$, *but* $F(z) \neq z$, *or*

c) $\rho \in E \setminus \{1\}$, *all* F_l *are iterable for large* l, *or*

d) F *is not iterable.*

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Then there exists an $l_0 \in \mathbb{N}$ *and a sequence* $(S_l)_{l \geq l_0}$ *of power series transformations* $S_l(z) = z + \cdots$ *such that* $\lim_{l\to\infty} S_l$ *exists and* $F_l = S_l^{-1} \circ F \circ S_l$ *for all* $l \geq l_0$. ■

3. The Local Distribution of Iterable Power Series Transformations in Higher Dimensions

The distribution problem (explained in Section 1) for automorphisms of $\mathbb{C}[[x_1, \ldots, x_n]]$ with $n \geq 2$ so far has only partial answers. The main reason for this is that the semi-canonical forms are not so well understood as for $n = 1$. Nevertheless some of our results may be worth mentioning. For the details about semi-canonical forms and about the iteration problem we again quote the survey papers [9], [10], [11], [16], where the reader may find references to the original articles.

Theorem 6. *Let* $F \in \Gamma$, $F(x) = Ax + \mathcal{P}(x)$, *where* $A \in GL(n, \mathbb{C})$, $\operatorname{ord}(\mathcal{P}) \geq 2$. *Denote by* $\rho_1, \ldots, \rho_n$ *the eigenvalues of* A. *Let* $\mathfrak{R}$ *be the set of all relations of the form*

$$\rho_k = \rho_1^{\nu_1} \ldots \rho_n^{\nu_n}$$

for $k = 1, \ldots, n$, $\nu \in \mathbb{N}_0^n$, $\nu_i \geq 0$, $|\nu| \geq 2$. *We assume that* $\mathfrak{R}$ *is finite (possibly empty). Then, if F is iterable, there is a neighbourhood U of F in the order topology such that each* $G \in U$ *is iterable. If F is noniterable, then there is a neighbourhood U of F such that each* $G \in U$ *is noniterable.*

Proof: According to Section 1 we may assume that F is in its semicanonical form. The finiteness of $\mathfrak{R}$ means that

$$F(x) = Jx + \mathcal{P}(x),$$

where J is in Jordan normal form and $\mathcal{P}(x)$ is a polynomial. More precisely, a monomial x^ν, $|\nu| \geq 2$, in the k-th component $\mathcal{P}_k(x)$ of $\mathcal{P}$ may have a nonzero coefficient $c_{k,\nu}$ only if

$$\rho_k = \rho_1^{\nu_1} \ldots \rho_n^{\nu_n}$$

holds. These monomials are called *additional (resonance) monomials for* ρ_k *with respect to* $\rho_1, \ldots, \rho_n$. If $N \in \mathbb{N}$ is sufficiently large, then each power series transformation $\tilde{F} \in U_N(F)$ has the form

$$\tilde{F}(x) = Jx + \mathcal{P}(x) + \tilde{\mathcal{R}}(x),$$

where $\operatorname{ord}(\tilde{\mathcal{R}}) > N (\geq \deg(P))$. It is well known in the theory of normal forms that we can obtain a semicanonical form of $\tilde{F}$ by the following procedure ('Formales Ausfegen', cf. [3]): There is a transformation $S \in \Gamma$, $S(x) = x + \mathcal{S}(x)$, with $\operatorname{ord}(\tilde{\mathcal{S}}) > N(\geq \deg(P))$, such that $\tilde{N} := S^{-1} \circ \tilde{F} \circ S$ is the semicanonical form of $\tilde{F}$ This S operates on the N-jet $Jx + \mathcal{P}(x)$ of $\tilde{F}$ as identity. Hence $\tilde{N}(x) = \tilde{F}(x)$, since there are no additional monomials of degree $> N$. Hence each $\tilde{F} \in U_N(F)$ is conjugate to F, and therefore iterable iff F is. This proves Theorem 6. ■

The assumption that the set $\mathfrak{R}$ of relations $\rho_k = \rho_1^{\nu_1} \cdots \rho_n^{\nu_n}$, $\nu_i \geq 0$, $|\nu| \geq 2$, be finite is fulfilled in the important special case of the socalled *contractions*, where $0 < |\rho_k| < 1$ for $k = 1, \ldots, n$, see [8].

If we do not make any assumption on the set $\mathfrak{R}$ of multiplicative relations for the eigenvalues, then we only can prove a result on sequences converging to F.

Theorem 7. *If $F \in \Gamma$ is iterable [noniterable], then there is a sequence $(F_{(k)})_{k \in \mathbb{N}}$ in Γ convergent to F such that each $F_{(k)}$ is iterable [noniterable] and $F_{(k)} \neq F$ for all k.*

Proof: We start with a lemma which will also be useful later.

Lemma 5. *If $F \in \Gamma$, $F(x) = Ax + \mathcal{P}(x)$, and if not all eigenvalues of A are equal to 1, then for each $N \in \mathbb{N}$ there exists $k \in [1, n]$ and $\nu \in \mathbb{N}_0^n$ such that $|\nu| > N$ and x^ν is not an additional monomial for ρ_k with respect to the eigenvalues $\rho_1, \ldots, \rho_n$ of A.*

Proof: If the assertion of Lemma 5 is false, then there is a number $N_0 \in \mathbb{N}$ such that for each $k \in [1, n]$ and each $\nu \in \mathbb{N}_0^n$ with $|\nu| > N_0$ the relation

$$\rho_k = \rho_1^{\nu_1} \cdots \rho_n^{\nu_n}$$

holds. In particular, if M is large enough, then we have

$$\rho_k = \rho_k^M$$

for each k, or $\rho_k^L = 1$ for each k and each sufficiently large L. Write $\rho_k = re^{2\pi i\alpha}$ with $r > 0$ and $0 \leq \alpha < 1$. Then $\alpha = a/b$, where $a, b \in \mathbb{Z}$, $b \geq 1$, $\gcd(a, b) = 1$, $r = 1$ and $La/b \in \mathbb{Z}$ for all sufficiently large L. We choose L so that $\gcd(L, b) = 1$, hence $\gcd(La, b) = 1$ which means $b = 1$, if $a \neq 0$, or $a = 0$, since $La/b \in \mathbb{Z}$. But since $a \in \mathbb{Z}$, $0 \leq a/b < 1$, only $a = 0$ is possible, and $\rho_k = 1$ for $k = 1, \ldots, n$ contradicting the assumption on the eigenvalues of A. This proves Lemma 5. ■

We turn now to the proof of Theorem 7. Assume first that all eigenvalues of A are equal to 1. Then each $G \in U_N(F)$ has linear part A and hence is iterable. Now assume that not all eigenvalues of A are equal to 1. Without loss of generality we take F as semicanonical form

$$F(x) = Jx + \mathcal{N}(x).$$

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Let $N \in \mathbb{N}$, $N \geq 1$. According to Lemma 5 there exists $k \in [1, n]$ and $\nu \in \mathbb{N}_0^n$ such that $|\nu| > N$ and x^ν is *not* an additional monomial for ρ_k. We fix these k and ν; moreover we may select them so that ν is minimal in the lexicographical ordering of monomials, and k is minimal with respect to the chosen $|\nu|$. Then write $F(x)$ as

$$F(x) = Jx + \mathcal{N}^*(x) + \mathcal{N}^{**}(x),$$

where $\mathcal{N}^*(x)$ consists of all additional monomials x^μ with $|\mu| < |\nu|$ (coefficients 0 admitted), while $\mathcal{N}^{**}(x)$ consists of the additional monomials x^μ with $|\mu| \geq |\nu|$, all in their appropriate place. Let S be the transformation

$$S(x) = x + \begin{pmatrix} \delta_{k1} \\ \\ \delta_{kk} \\ \\ \delta_{kn} \end{pmatrix} x^\nu$$

and form $G := S^{-1} \circ F \circ S$. Then S operates on the $(|\nu| - 1)$-jet $Jx + \mathcal{N}^*(x)$ of $F(x)$ as identity. A detailed calculation shows that the coefficient of x^ν in the k-th component $G_k(x)$ of G is $\rho_1^{\nu_1} \cdots \rho_n^{\nu_n} - \rho_k$; hence $\neq 0$, since x^ν is not an additional monomial for ρ_k. We omit the details here, but indicate only that the minimality of k and ν is crucial in the arguments, as well as the fact that substitution of additional monomials into additional monomials yields only additional monomials (see [8]).

Summarizing, we obtain $G \in U_N(F)$, $G \neq F$, and G is conjugate to F. Hence G is iterable iff F is iterable. Since N was arbitrary, Theorem 7 is proved. ■

If the semicanonical form $F \in \Gamma$ is not linearizable, then F has a *minimal* additional monomial (with a nonzero coefficient) which is an invariant of the conjugacy class of F. This minimal additional monomial x^{ν_0} is defined as follows:

1) $|\nu| =: m_0$ is minimal for all additional monomials x^μ which appear in F with nonzero coefficient.
2) Then take the minimal index k for which x^ν with $|\nu| = m_0$ is an additional monomial for ρ_k, with nonzero coefficient in F.
3) Among those x^ν satisfying 1) and 2) take the monomial x^{ν_0} which is minimal in the lexicographical order having a nonzero coefficient in F.

This minimal additional monomial is the same for all semicanonical forms of F. Furthermore (see [4], [5]), we know that $F \in \Gamma$ has an analytic iteration iff it is iterable at all. From the theory of analytic iterations (see [9], [10], [16]) we know that each analytic iteration of F is associated with a certain choice $\Lambda = (\ln \rho_1, \ldots, \ln \rho_n)$ of the logarithms of the eigenvalues $\rho_1, \ldots, \rho_n$ of A in $F(x) = Ax + \mathcal{P}(x)$, and F has an analytic iteration with respect to a given Λ iff it has a socalled *smooth* normal form with respect to Λ. Using these notions we can prove

Theorem 8. *Let $F \in \Gamma$ be not linearizable and assume that the minimal additional monomial of F is not smooth with respect to any choice $\Lambda = (\ln \rho_1, \ldots, \ln \rho_n)$ of the logarithms of the eigenvalues $\rho_1, \ldots, \rho_n$ of F. Then there is a neighbourhood of F which contains only noniterable automorphisms.*

Proof: We may assume that F is already a semicanonical form, and that $N \geq 2$ is the degree of its minimal additional monomial. Let $G \in U_N(F)$. Then G has the same N-jet as F, and hence the structure of a semicanonical form mod ord N. We know already that there is a transformation $S \in \Gamma$ acting on the N-jet of G as identity and transforming G into a semicanonical form H. Assume that G is analytically iterable. Then it has a smooth normal form $\tilde{H}$ with respect to a certain choice Λ^0 of the logarithms. But $\tilde{H}$ and H have the same minimal additional monomial, and obviously H and F have the same minimal additional monomial, too. So this would be smooth with respect to Λ^0, a contradiction. ■

Theorem 1(iv) is a special case of Theorem 8, as can easily be checked.

One may ask, whether a power series transformation $F \in \Gamma$, surrounded by a large enough set of iterable [noniterable] transformations is iterable [noniterable] itself. A (rather weak) answer to this equation is

Theorem 9. *Let $F \in \Gamma$, and $U(F)$ be a strong neighbourhood of F such that each $G \in U(F) \setminus \{F\}$ is iterable [noniterable]. Then F is iterable [noniterable] itself.*

Proof: If the linear part of F has only 1 as eigenvalue, then F is iterable. So assume that $F(x) = Ax + \mathcal{P}(x)$, where A has an eigenvalue different from 1. Then, according to Lemma 5, for each $N \in \mathbb{N}$ there is $k \in [1, n]$ and $\nu \in \mathbb{N}_0^n$ with $|\nu| > N$ such that x^ν is not an additional monomial for ρ_k ($\rho_1, \ldots, \rho_n$ being the eigenvalues of A). Then the argument in the proof of Theorem 7 gives us $S \in \Gamma$ such that $S^{-1} \circ F \circ S \in U_N(F)$ and $S^{-1} \circ F \circ S \neq (F)$. Hence $G := S^{-1} \circ F \circ S$ is iterable iff F is iterable. This finishes the proof. ■

Here are some possibilities to weaken the assumptions of Theorem 9. E.g., in order to deduce the iterability of F it is sufficient to assume

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the existence of a countable set $\mathcal{C}$ in $U_N(F)$ such that all $G \in U_N(F)\backslash(\{F\} \cup \mathcal{C})$ are iterable. Another possibility is to assume the iterability of all G in $U_N(F)\backslash(\{F\} \cup \mathcal{D})$, where $\mathcal{D}$ is defined as the set of all $H \in U_N(F)$ such that for each semicanonical form N of F the conjugate $H^{-1} \circ N \circ H$ also is a semicanonical form. The set $\mathcal{D}$ can be uncountable, though still 'thin' and of a very special nature. We omit the proofs of these remarks.

The information about the structure of convergent sequences in the higher-dimensional case is also not very satisfactory. Our technique of semicanonical forms yields

Theorem 10. *Let* $(F_{(l)})_{l\in\mathbb{N}}$ *be a convergent sequence in* Γ *with* $F = \lim_{l\to\infty} F_{(l)}$. *Then there is* $l_0 \in \mathbb{N}$ *such that for* $l \geq l_0$ *there is a transformation* $T_{(l)} \in \Gamma$, $T_{(l)}(x) = x + \cdots$, *such that*

a) $\lim_{l\to\infty} T_{(l)} = T$ *exists,*
b) $T_{(l)}^{-1} \circ F_{(l)} \circ T_{(l)}$ *is a semicanonical form for* $l \geq l_0$, *and*
c) $T^{-1} \circ F \circ T$ *is a semicanonical form.*

Proof: Finding a semicanonical form N of F means solving the functional equation $F \circ T = T \circ N$ for T and N, where we have the conditions

1) $T(x) = x + \cdots$, and
2) $N(x) = Jx + \mathcal{N}(x)$, where J is a Jordan normal form and in the k-th component $\mathcal{N}_k(x)$ of $\mathcal{N}(x)$ a monomial x^ν can have a nonzero coefficient $d_{k,\nu}$ only if the relation $\rho_k = \rho_1^{\nu_1} \cdots \rho_n^{\nu_n}$ holds, where $\rho_1 \ldots \rho_n$ are the eigenvalues of J, $\nu = (\nu_1, \ldots, \nu_n)$, $|\nu| \geq 2$.

This Schröder type equation always has a solution, but in general the solution is not unique. We can enforce uniqueness, if we require (see [8]) that the coefficient $t_{k,\nu}$ of x^ν in the kth component of $T(x) = x + \cdots$ be 0 if $\rho_k = \rho_1^{\nu_1} \cdots \rho_n^{\nu_n}$ holds (i.e. if x^ν is an additional monomial for ρ_k). Doing so, we find that each coefficient $t_{k,\mu}$ is a polynomial in the coefficients $a_{l,\lambda}$ of F with $l = 1, \ldots, n$, $|\lambda| \leq |\nu|$, being rational in $\rho_1, \ldots, \rho_n$. The coefficients of the semicanonical form N are then also uniquely determined, in fact, the coefficient $g_{k,\nu}$ of an additional monomial x^ν in the component $\mathcal{N}_k(x)$ is a polynomial in the coefficients $a_{l,\lambda}$ of F with $l = 1, \ldots, n$ and $|\lambda| \leq |\nu|$. Applying this construction to F and $F_{(l)}$ for $l \geq l_0$ (for which $F_{(l)} \in U_1(F)$), we find transformations $T_{(l)}$, $l \geq l_0$, of $F_{(l)}$ to a semicanonical form $N_{(l)}$ such that $\lim_{l\to\infty} T_{(l)} = T$ and $\lim_{l\to\infty} N_{(l)} = N$ exist, N is a semicanonical form of F, and the Theorem is proved. ∎

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