A Convergence Lemma for the Parry–Daniels Map

By

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Abstract

Nogueira has shown that the 2-dimensional Parry–Daniels map is ergodic. The proof uses the fact that appropriate sequences of cylinders shrink to points. The purpose of this note is to give a proof of this property which shows how the appearance of different types of digits is related to the convergence rate.

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Let \( \Sigma = \{x = (x_0, x_1, x_2) \mid 0 \leq x_0, x_1, x_2 \leq 1, x_0 + x_1 + x_2 = 1\} \). The 2-dimensional Parry–Daniels map \( T : \Sigma \to \Sigma \) is defined as follows. Let \( \pi \) be a permutation such \( x_{\pi 0} \leq x_{\pi 1} \leq x_{\pi 2} \) then

\[
T(x_0, x_1, x_2) = \left( \frac{x_{\pi 0}}{x_{\pi 2}}, \frac{x_{\pi 1} - x_{\pi 0}}{x_{\pi 2}}, \frac{x_{\pi 2} - x_{\pi 1}}{x_{\pi 2}} \right).
\]

Then \((\Sigma, T)\) is a fibred system (Schweiger 1995, 2000) with the time-1-partition \( B(\pi) = \{x \in \Sigma \mid \pi = \pi(x)\} \). The digits are the six permutations \( \varepsilon, (01), (02), (12), (021), (012) \). As usual we put \( \pi_j = \pi(T^{j-1}x) \),
It is well known that $T$ admits an invariant measure with density

$$h(x) = \frac{1}{x_0(x_0 + x_1)}$$

Daniels (1962) asked if $T$ is ergodic with respect to Lebesgue measure. Parry (1962) proved that the 1-dimensional Parry–Daniels map is ergodic. A further step in this direction was given in Schweiger (1981). Let

$$\Gamma = \{x \in \Sigma \; \pi_j(x) = \varepsilon \text{ oder } (01) \text{ for any } j = 1, 2, \ldots \}$$

then $\lambda(\Gamma) > 0$. Therefore $T$ is not conservative. It was reasonable to conjecture that $\Gamma$ is an absorbing set (i.e. for almost all $x \in \Sigma$ there is an $n = n(x)$ such that $T^n x \in \Gamma$). Nogueira (1995) eventually showed that in fact, $\Gamma$ is an absorbing set and $T$ is ergodic. The proof uses the following result.

**Theorem 1. ("Shrinking Lemma").** Let $\pi_s(x) \in \{(012), (021), (02), (12)\}$ for infinitely many values of $s$ then $\lim_{s \to \infty} \text{diam} \; B(\pi_1, \ldots, \pi_s) = 0$.

We first point out that without an additional condition on the digits $\pi_j, j = 1, 2, \ldots$ the Shrinking Lemma is not generally true. We consider $(\alpha, \beta, \gamma) \in \Sigma, \alpha = -2 + \sqrt{5}, \beta = \frac{7 - 3\sqrt{5}}{2}, \gamma = \frac{-1 + \sqrt{5}}{2}$. Then $(\alpha, \beta, \gamma)$ is a fixed point for $T$. Therefore the segment $\lambda(\alpha, \beta, \gamma) + (1 - \lambda)(0, 0, 1), 0 \leq \lambda \leq 1$, is invariant under $T$. This shows that

$$\text{diam} \; B((01), (01)) \geq 2\sqrt{5} - 4 > 0.$$
If

\[ M(\pi_1) \cdot M(\pi_s) := \begin{pmatrix} a_{s0} & b_{s0} & c_{s0} \\ a_{s1} & b_{s1} & c_{s1} \\ a_{s2} & b_{s2} & c_{s2} \end{pmatrix} \]

and

\[ A_s := a_{s0} + a_{s1} + a_{s2}, \quad B_s := b_{s0} + b_{s1} + b_{s2}, \quad C_s := c_{s0} + c_{s1} + c_{s2} \]

then the map

\[ V_s = V(\pi_1, \ldots, \pi_s) : \Sigma \to \Sigma(\pi_1, \ldots, \pi_s), \quad V_s x = y \]

maps the simplex \( \Sigma \) onto the cylinder \( \Sigma(\pi_1, \ldots, \pi_s) \).

We put \( \Delta(\pi_1, \ldots, \pi_s) := \text{diam} \, \Sigma(\pi_1, \ldots, \pi_s) \). The following lemma is straightforward.

**Lemma 1.** Let \( x = (x_0, x_1, x_2), y = (y_0, y_1, y_2), z = (z_0, z_1, z_2) \) be three collinear points, \( z = \lambda x + \mu y \), say. Then

\[
\frac{d(V_s x, V_s z)}{d(V_s x, V_s y)} = |\mu| \left| \frac{A_s y_0 + B_s y_1 + C_s y_2}{A_s z_0 + B_s z_1 + C_s z_2} \right|.
\]

**Lemma 2.** Let \( J_0 = V_s(1, 0, 0), \quad J_1 = V_s(0, 1, 0), \quad J_2 = V_s(0, 0, 1), \quad M_1 = V_s(\frac{1}{2}, 0, \frac{1}{2}), \quad M_2 = V_s(\frac{1}{2}, \frac{1}{2}, 0), \quad M_0 = V_s(0, \frac{1}{2}, \frac{1}{2}), \quad Z = V_s(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}) \) then we find the following ratios.

\[
\begin{align*}
\frac{d(J_2, M_1)}{d(J_0, J_2)} &= \frac{A_s}{A_s + C_s}, & \frac{d(J_0, M_1)}{d(J_0, J_2)} &= \frac{C_s}{A_s + C_s} \\
\frac{d(J_0, M_2)}{d(J_0, M_2)} &= \frac{B_s}{B_s + A_s}, & \frac{d(J_1, M_2)}{d(J_1, J_0)} &= \frac{A_s}{B_s + A_s} \\
\frac{d(J_1, M_0)}{d(J_1, J_1)} &= \frac{C_s}{C_s + B_s}, & \frac{d(J_2, M_0)}{d(J_2, J_1)} &= \frac{B_s}{C_s + B_s} \\
\frac{d(J_0, Z)}{d(J_0, M_0)} &= \frac{B_s + C_s}{A_s + B_s + C_s}, & \frac{d(M_0, Z)}{d(J_0, M_0)} &= \frac{A_s}{A_s + B_s + C_s} \\
\frac{d(J_1, Z)}{d(J_1, M_1)} &= \frac{C_s + A_s}{A_s + B_s + C_s}, & \frac{d(M_1, Z)}{d(J_1, M_1)} &= \frac{B_s}{A_s + B_s + C_s}
\end{align*}
\]
\[
\frac{d(J_2, Z)}{d(J_2, M_2)} = \frac{A_s + B_s}{A_s + B_s + C_s}, \quad \frac{d(M_2, Z)}{d(J_2, M_2)} = \frac{C_s}{A_s + B_s + C_s}
\]

**Lemma 3.** If \( \pi_{s+1} = (02) \) or \((012)\) then
\[
\Delta(\pi_1, \ldots, \pi_s, \pi_{s+1}) \leq \frac{3}{4} \Delta(\pi_1, \ldots, \pi_s).
\]

*Proof:* Observe that \( M_1, Z, J_0 \) are the vertices of \( \Sigma(\pi_1, \ldots, \pi_s, (012)) \) and \( J_0, Z, M_2 \) are the vertices of \( \Sigma(\pi_1, \ldots, \pi_s, (02)) \). Since for all \( s \geq 1 \) the inequalities
\[
C_s \leq B_s \leq A_s
\]
are valid we see that
\[
\max \left( \frac{C_s}{A_s + C_s}, \frac{B_s}{A_s + B_s}, \frac{B_s}{A_{s+1} + B_s}, \frac{C_s}{A_{s+1} + C_s}, \frac{B_s + C_s}{A_{s+1}} \right) \leq \frac{2}{3}.
\]

**Lemma 4.** If \( \pi_{s+1} = (021) \) or \((12)\) and \( \pi_s \in \{(01), (012), (021), (02)\} \) then
\[
\Delta(\pi_1, \ldots, \pi_s, \pi_{s+1}) \leq \frac{3}{4} \Delta(\pi_1, \ldots, \pi_s).
\]

*Proof:* The vertices of \( \Sigma(\pi_1, \ldots, \pi_s, (021)) \) are \( J_1, Z, M_2 \) and the vertices \( \Sigma(\pi_1, \ldots, \pi_s, (12)) \) are \( M_0, Z, J_1 \). An inspection of the list of ratios shows that the ratio
\[
\frac{d(J_1, M_0)}{d(J_1, J_2)} = \frac{C_s}{B_s + C_s} \leq \frac{1}{2}
\]
is not problematic. A more careful analysis is required for the other ratios. We illustrate the method for \( \pi_s = (012), \pi_{s+1} = (021) \). Then
\[
A_s \leq A_{s-1} + B_{s-1} + C_{s-1} < \frac{2}{3}, \quad A_s + C_s \leq 2A_{s-1} + B_{s-1} + 2C_{s-1} < \frac{3}{4}.
\]

**Lemma 5.** Let \( \pi_{s+1} = (021) \) or \((12)\).

Assume \( \pi_j \in \{\varepsilon, (12)\} \) for \( s - w + 1 \leq j \leq s \) but \( \pi_{s-w} \in \{(01), (012), (021), (02)\} \). Then
\[
\Delta(\pi_1, \ldots, \pi_s, \pi_{s+1}) \leq \frac{2}{3} \Delta(\pi_1, \ldots, \pi_{s-w}).
\]

*Proof:* If \( \pi_{s-w} \in \{(01), (012), (021), (02)\} \) then \( A_{s-w} \leq 2B_{s-w} \).
We note that the cylinder $\Sigma(\varepsilon, \varepsilon) = \{x \mid \pi_j(x) = \varepsilon, 1 \leq j \leq w\}$ has the vertices $(0, 0, 1)$, $Q_w = \left(0, \frac{1}{w+1}, \frac{w}{w+1}\right)$, and

$$R_w = \left(\frac{2}{(w+1)(w+2)}, \frac{2w}{(w+1)(w+2)}, \frac{w}{w+2}\right).$$

Then the cylinder $\Sigma(\varepsilon, \varepsilon, (12))$ has the vertices $Q_w, Q_{w+1}$, and $R_{w+1}$. The cylinder $\Sigma(\varepsilon, \varepsilon, (021))$ has the vertices $Q_w, R_{w+1}$, and

$$S_w = \left(\frac{2}{(w+1)(w+4)}, \frac{2}{w+4}, \frac{w(w+3)}{(w+1)(w+4)}\right).$$

We apply Lemma 1 to the map $V_{s-w} : \Sigma \to \Sigma(\pi_1, \ldots, \pi_{s-w})$.

Now assume $\pi_{s+1} = (021)$ or $(12)$. Let $\pi_{s-w} \in \{(012), (02), (021), (01)\}$ and $\pi_j \in \{\varepsilon, (12)\}, s-w+1 \leq j \leq s$. We first consider the case $\pi_j = \varepsilon, s-w+1 \leq j \leq s$.

Let $w = 1$. Then the line through $Q_1 = (0, \frac{1}{2}, \frac{1}{2})$ and $R_2 = (\frac{1}{6}, \frac{2}{6}, \frac{3}{6})$ meets $x_1 = 0$ in the point $T_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$. Therefore

$$d(Q_1, R_2) = \frac{A_{s-1} + C_{s-1}}{A_{s-1} + 2B_{s-1} + 3C_{s-1}} \leq \frac{1}{2}$$

since $A_{s-1} \leq 2B_{s-1}$. We also find

$$d(Q_1, S_1) = \frac{A_{s-1}}{A_{s-1} + 2B_{s-1} + 2C_{s-1}} \leq \frac{1}{2}$$

and

$$d(S_1, R_2) = \frac{C_{s-1}}{A_{s-1} + 2B_{s-1} + 3C_{s-1}} \leq \frac{1}{6}.$$ 

The line through $Q_2 = (0, \frac{1}{3}, \frac{2}{3})$ and $R_2 = (\frac{1}{6}, \frac{2}{6}, \frac{3}{6})$ meets $x_2 = 0$ in the point $Y_2 = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$. Therefore, again

$$d(Q_2, R_2) = \frac{2A_{s-1} + B_{s-1}}{2A_{s-1} + 4B_{s-1} + 6C_{s-1}} \leq \frac{2}{3}.$$

Clearly

$$d(Q_1, Q_2) = \frac{1}{d(Q_1, J_2)} = \frac{1}{3}.$$
Now consider $w \geq 2$. The line through $Q_w = \left(0, \frac{1}{w+1}, \frac{w}{w+1}\right)$ and $R_{w+1} = \left(\frac{2}{(w+2)(w+3)}, \frac{2(w+1)}{(w+2)(w+3)}, \frac{w+1}{w+3}\right)$ meets $x_2 = 0(1)$ in the point $T_w = \left(\frac{2w^2-w-2}{(w+2)(w-1)}, \frac{w}{w+2}(w-1), 0\right)$. Then we find
\[
\frac{d(Q_w, R_{w+1})}{d(Q_w, T_w)} \leq \frac{2wA_{s-w} + (w^2 - w - 2)B_{s-w}}{2wA_{s-w} + (2w^2 + 2w)B_{s-w}} \leq \frac{1}{2}.
\]
Furthermore, the line through $Q_{w+1} = \left(0, \frac{1}{w+2}, \frac{w+1}{w+2}\right)$ and $R_{w+1} = \left(\frac{2}{(w+2)(w+3)}, \frac{2(w+1)}{(w+2)(w+3)}, \frac{w+1}{w+3}\right)$ meets $x_2 = 0$ in the point $Y_{w+1} = \left(\frac{2}{w+2}, \frac{w}{w+2}, 0\right)$. Then
\[
\frac{d(Q_{w+1}, R_{w+1})}{d(Q_{w+1}, Y_{w+1})} \leq \frac{2A_{s-w} + wB_{s-w}}{2A_{s-w} + (2w + 2)B_{s-w}} \leq \frac{2}{3}.
\]
Last but not least we find
\[
\frac{d(Q_w, Q_{w+1})}{d(Q_w, J_2)} = \frac{1}{w + 2} \leq \frac{1}{4}.
\]
The estimate
\[
\frac{d(S_w, R_{w+1})}{d(S_w, J_2)} = \frac{2C_{s-1}}{2A_{s-1} + 2(w + 1)B_{s-1} + (w + 1)(w + 2)C_{s-1}} \leq \frac{2}{(w + 2)(w + 3)} \leq \frac{1}{6}
\]
is unconditionally true.

The line through $Q_w$ and $R_w$ meets $x_2 = 0$ in the point $Y_w = \left(\frac{2}{w+1}, \frac{w-1}{w+1}, 0\right)$ and we find as before
\[
\frac{d(Q_w, S_w)}{d(Q_w, Y_w)} \leq \frac{d(Q_w, R_w)}{d(Q_w, Y_w)} \leq \frac{2}{3}.
\]
In the other case let $\pi_j = \varepsilon, s - w + 1 \leq j \leq t$ and $\pi_{t+1} = (12)$, $t + 1 \leq s$. Since clearly
\[
\Delta(\pi_1, \ldots, \pi_{s+1}) \leq \Delta(\pi_1, \ldots, \pi_{s+1})
\]
we are back to a situation already considered and we find
\[
\Delta(\pi_1, \ldots, \pi_{s-1}) \leq \frac{2}{3} \Delta(\pi_1, \ldots, \pi_{s-w}).
\]
The only remaining case is $\pi_j \in \{\varepsilon, (12)\}$ for all $j \geq j_0$. 
Lemma 6. Let \( \pi_j \in \{\varepsilon, (12)\} \) for all \( j \geq 1 \). Then

\[
\Delta(\pi_1, \ldots, \pi_s) \ll \frac{1}{\sqrt{s}}.
\]

Proof: Let \( J_0 = \left( a_{00}, a_{01}, a_{12} \right), J_1 = \left( b_{00}, b_{01}, b_{12} \right), \) and \( J_2 = \left( c_{00}, c_{01}, c_{12} \right) \) be the three vertices of \( B(\pi_1, \ldots, \pi_s) \). Note that \( b_{s0} = c_{s0} = 0 \) but this fact does not help very much. To estimate the distances \( d(J_0, J_1), d(J_1, J_2), \) and \( d(J_2, J_0) \) we need estimates for the determinants

\[
\begin{align*}
[A, B]^j_s & := \begin{vmatrix}
 a_{sj} & b_{sj} \\
 A_s & B_s
\end{vmatrix}, \\
[B, C]^j_s & := \begin{vmatrix}
 b_{sj} & c_{sj} \\
 B_s & C_s
\end{vmatrix}, \quad j = 0, 1, 2. \\
[C, A]^j_s & := \begin{vmatrix}
 c_{sj} & a_{sj} \\
 C_s & A_s
\end{vmatrix},
\end{align*}
\]

Since the estimate is the same for \( j = 0, 1, 2 \) we may omit the index \( j \).

Note that for \( \pi_{s+1} = \varepsilon \) or \((12)\)

\[
\begin{align*}
A_{s+1} &= A_s + B_s + C_s, \\
B_{s+1} &= B_s + C_s
\end{align*}
\]

but

\[
\begin{align*}
C_{s+1} &= C_s \quad \text{if} \quad \pi_{s+1} = \varepsilon \\
C_{s+1} &= B_s \quad \text{if} \quad \pi_{s+1} = (12).
\end{align*}
\]

Since \( A_1 = 3, B_1 = 2, C_1 = 1, \) we get \( A_s \leq (s+1)B_s \).

Since \( |[B, C]_s| = |[B, C]_s| \) we find by induction that \( |[B, C]_s| \leq 1 \).

This shows

\[
\frac{|[B, C]_s|}{B_s C_s} \leq \frac{1}{B_s} \leq \frac{1}{s + 1}.
\]

Put \( \theta(s) = \max \left( \frac{|[A, B]_s|}{A_s B_s}, \frac{|[C, A]_s|}{A_s C_s} \right) \). Then

\[
\begin{align*}
\frac{|[A, B]_{s+1}|}{A_{s+1} B_{s+1}} & \leq \frac{|[A, B]_s| + |[C, A]_s|}{A_{s+1}(B_s + C_s)} \\
& \leq \theta(s) \frac{A_s}{A_s + B_s + C_s} \leq \theta(s) \frac{s + 1}{s + 2}.
\end{align*}
\]
Now assume that $\pi_{s+1} = (12)$. Then
\[
\frac{||[C,A]_{s+1}||}{A_{s+1}C_{s+1}} \leq \frac{||[A,B]_s|| + ||[B,C]_s||}{A_{s+1}B_s} \leq \max \left( \left( \frac{||[A,B]_s||}{A_s B_s}, \frac{||[B,C]_s||}{C_s B_s} \right) \right) \leq \max \left( \theta(s) \frac{s + 1}{s + 2} \right) .
\]

If $\pi_{s+1} = \varepsilon$ then we find
\[
\frac{||[C,A]_{s+1}||}{A_{s+1}C_{s+1}} \leq \frac{||[C,A]_s|| + ||[B,C]_s||}{A_{s+1}C_s} .
\]

If $\pi_s = \varepsilon$ we expand further and find
\[
\frac{||[C,A]_{s+1}||}{A_{s+1}C_{s+1}} \leq \frac{||[C,A]_{s-1}|| + 2||[B,C]_{s-1}||}{(A_{s-1} + 2B_{s-1} + 3C_{s-1})C_{s-1}} \leq \max \left( \left( \frac{||[C,A]_{s-1}||}{A_s C_{s-1}}, \frac{2||[B,C]_{s-1}||}{B_{s-1}C_{s-1} + 2C^2_{s-1}} \right) \right) \leq \max \left( \theta(s - 1) \frac{s}{s + 1}, \frac{2}{B_{s-1} + 2} \right) \leq \max \left( \theta(s - 1) \frac{s}{s + 1}, \frac{2}{s + 2} \right) .
\]

If $\pi_s = (12)$ we obtain
\[
\frac{||[C,A]_{s+1}||}{A_{s+1}C_{s+1}} \leq \frac{||[A,B]_{s-1}|| + 2||[B,C]_{s-1}||}{(A_{s-1} + 3B_{s-1} + 2C_{s-1})B_{s-1}} \leq \max \left( \left( \frac{||[A,B]_{s-1}||}{A_s B_{s-1}}, \frac{2||[B,C]_{s-1}||}{2B^2_{s-1} + B_{s-1}C_{s-1}} \right) \right) \leq \max \left( \theta(s - 1) \frac{s}{s + 1}, \frac{2}{s + 2} \right) .
\]

Hence $\theta(s + 1) \leq \max \left( \theta(s - 1) \frac{s}{s + 1}, \theta(s) \frac{s + 1}{s + 2}, \frac{2}{s + 2} \right)$. It is easy to see that $\theta(s) \leq \frac{1}{\sqrt{s+1}}$ satisfies these recursive conditions.

**Remark:** In fact one can show that
\[
E = \{ x \in \Sigma \mid \pi_j(x) = \varepsilon \lor (12) \text{ for all } j \geq 1 \} = \{ x \in \Sigma \mid x_0 = 0 \}.
\]
This follows from the fact that $0 \leq x_0 \leq \frac{1}{s}$ implies

$$(V(\varepsilon)x)_0 = (V(01)x)_0 = \frac{x_0}{1 + 2x_0 + x_1} \leq \frac{1}{s + 1}.$$ 

References


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