Johannis (I) Bernoullii Lectiones de calculo differentialium.

Unter Mithilfe der Familie Bernoulli herausgegeben von der Naturforschenden Gesellschaft in Basel, 300 Jahre nach der Aufnahme der Bernoulli ins Basler Bürgerrecht (13. Mai 1622).

Mit einem Vorwort
von
Paul Schafheitlin.

Hiezu 4 Tafeln (I—IV).

Vorwort.

Im vorletzten Bande dieser Zeitschrift[1]) *habe ich darauf aufmerksam gemacht, dass eine Handschrift der verloren geglaubten Differentialrechnung Joh. Bernoullis (1667—1748) in der Basler Universitätsbibliothek sich befindet und aus deren Vergleich mit des Marquis de l'Hospital „Analyse des infiniment petits" die Folgerung gezogen, dass die Differentialrechnung Bernoullis den ersten vier Abschnitten von Hospitals Analyse als Muster gedient hat. Damit aber die mathematische Welt selbst imstande ist, sich hierüber ein Urteil zu bilden, erscheint hier jene Handschrift ausführlich und wortgetreu, ohne dass an der vielfach schwerfälligen und häufig unklaren Formelschreibweise etwas geändert wird; in Fussnoten habe ich z. T. die Formeln in moderner Fassung wiederholt. Nur die Figuren, die bei Bernoulli im Texte sich befinden, sind der bequemeren Herstellung wegen in 4 faksimilierten Tafeln vereinigt und, mit Nummern versehen, am Schlusse angefügt worden; auf der letzten Tafel ist eine Seite der Handschrift als Faksimile gegeben worden.*

Im letzten Absatz obiger Notiz habe ich bemerkt, dass die Handschrift kein Datum trägt, so dass also mit dem Einwurf gerechnet werden muss, dass umgekehrt Bernoullis Differentialrechnung nach dem Vorbilde Hospitals angefertigt worden ist. Auf die Unwahrscheinlichkeit dieser Möglichkeit habe ich schon hingewiesen; ich möchte mir gestatten, hier noch auf einige Punkte aufmerksam zu machen, die für mich die Priorität Bernoullis zur Gewissheit erheben.

---

[1]) *Bd. 32, Seite 230—235.*

Erstens ist der Ausdruck und die Formelschreibung Bernoullis an vielen Stellen erheblich schwerfälliger und ungelenker als die Hospitals; ich weise z. B. auf Seite 5 der Handschrift hin und auf die Schreibart höherer Potenzen von Wurzeln usw. durch Vorsetzung der Zeichen ❏, C, QQ.

Zweitens tritt mehrfach bei der Handschrift eine geniale Unachtsamkeit zutage, wodurch Ungenauigkeiten und Flüchtigkeitsfehler hervorgerufen sind, die Hospital verbessert hat. Die zweite Art, das Problem 19, Seite 24 zu lösen, führt auf eine Gleichung dritten Grades, die Bernoulli ungelöst stehen lässt, da er offenbar übersehen hat, dass sie die einfache Wurzel $x = b$ zulässt, während Hospital (Nr. 60, Seite 51) dies erkannt hat und demgemäss mit Hilfe einer Gleichung zweiten Grades die Aufgabe löst. Bei Problem 21, Seite 30 sucht Bernoulli den Wendepunkt der Kurve, die jetzt als Versiera bezeichnet wird; die im Texte dabei gezeichnete Figur ist aber falsch, während Hospital sie richtig entworfen hat (Nr. 68, Fig. 58). Bei der letzten Aufgabe, den Wendepunkt der parabolischen Spirale zu bestimmen (Seite 38), haben die beiden letzten Glieder der Gleichung fünften Grades, auf die die Aufgabe führt, das falsche Vorzeichen, während Hospital (Nr. 73, Seite 68) die Gleichung richtig angibt.

Drittens möchte ich noch darauf hinweisen, dass in dem Heft, das die Differentialrechnung enthält, unmittelbar darauf ein Teil der Integralrechnung folgt, die bekanntlich in Bernoullis gedruckten Werken[1]) erschienen ist; hier ist die erste Seite [39] davon mitabgedruckt, die gegen die Fassung in den gesammelten Werken nur geringe Abweichungen aufweist. Aus ihr geht hervor, dass Bernoullis Abhandlung vor dem Jahre 1694, also vor dem Erscheinen der Analyse (1696), abgefasst sein muss. Denn dort findet sich die unrichtige Angabe, dass das Integral von $dx:x$ unendlich ist, während er 1694 den Wert des Integrals richtig als $\log x$ angegeben hat.[2])

Schliesslich ist noch ein kleiner Umstand zu erwähnen. Von den drei anfangs angeführten Postulaten Bernoullis übernimmt Hospital in wörtlicher französischer Übersetzung die beiden ersten, nicht aber das dritte; denn in der Differentialrechnung werden nur die beiden ersten, das dritte erst in der Integralrechnung gebraucht. Als ein zusammenhängendes Ganze schwebten offenbar Bernoullis Geiste beide Rechnungen vor, deren notwendige Voraussetzungen im Gegensatz zu den bisherigen Rechnungsarten dem Leser gleich zu Anfang klar gemacht werden müssen. Hospital betrachtet die Integralrechnung als eine Fortsetzung, die er ausdrücklich von seiner

---

[1]) *Opera omnia Band 3, Seite 385.*

[2]) *Acta Eruditorum 1694, Seite 437 ff.*

Betrachtung ausschliesst, und demnach lässt er absichtlich jenes dritte Postulat weg.

In dem von Bernoulli selbst verfassten Abriss seines Lebens erwähnt er, dass die Untersuchungen über Infinitesimalrechnung, die von ihm dem Marquis im Winter 1691/92 vorgetragen worden sind, von einem Freunde aufgeschrieben wurden. Um Missverständnissen vorzubeugen, bemerke ich, dass die aufgefundene Schrift von der Hand Nicolaus (I) Bernoulli herrührt, der 1687 geboren wurde. Unmöglich also kann die vorliegende Schrift jene Niederschrift aus dem Jahre 1691 sein. Nicolaus besuchte seinen Onkel Johann 1705 in Gröningen, um sich von ihm in die Mathematik einführen zu lassen und ich nehme an, dass er damals diese Abschrift von jenen Vorlesungen Johanns angefertigt hat; hätte er mündliche Unterweisungen Johanns zu Papier gebracht, so würde die Handschrift anders ausgefallen sein, z. B. jene falsche Ansicht über das Integral von $dx : x$ hätte Johann 1705 nicht mehr äussern können.

DE CALCULO DIFFERENTIALIUM.

[1]*) Postulata.

1. Quantitas diminuta vel aucta quantitate infinities minore neque diminuitur neque augetur.

2. Quaevis linea Curva constat ex infinitis rectis, iisque infinite parvis.

3. Figura contenta sub duabus ordinatim applicatis, differentia abscissarum, et portione infinite parva alicujus Curvae, consideratur ut Parallelogrammum.

De Differentialium Additione et Subtractione.

Regula 1. Quantitatum additarum differentialis est summa differentialium cujusque quantitatis, membrum additarum facientis, separatim sumptae.

Ex. gr. Quantitatis $x+y$ differentialis est $dx+dy$. Sit enim $e=dx=$ differentiae indeterminatae x et $f=dy=$ differentiae indeterminatae y. Addantur major $x+e$, et major $y+f$. Summa erit $x+y+e+f$, de qua si subtrahatur summa minorum $x+y$, restat differentia $e+f=dx+dy$. Q. E. D.

Differentialis quantitatis $a+x$ est dx; si a quantitatem certam et determinatam denotet, ut supponimus hic et in seqq.

---

*) Die in eckigen Klammern [ ] beigesetzten Zahlen bedeuten die Seiten der Handschrift.

Etenim addantur $a+0$ et $x+e$, summa erit $a+x+e$. Minor subtrahatur nempe $a+x$. Residuum erit $+e=dx$. Q. E. D.

Quae de Quantitatibus additis dicta sunt, mutatis mutandis, ad quantitates de se invicem subtractas etiam applicari possunt.

[2] De Quantitatum compositarum Differentialibus.

Quantitatis ax differentialis est $a\,dx$. Quod sic probatur:

multipl. $x+e$ suppone $e=dx$
cum $a+0$ id est a plus nihil, quia a est quantitas
prod. $ax+ae$ determinata, quae nullam habet differentialem
de quo subtr. ax
restat $ae=a\,dx$ q. e. d.

Quantitatis xx differentialis est $2x\,dx$. quod sic demonstratur: Multiplicata $x+e$ per $x+e$. Productum erit $xx+2ex+ee$. Subtrahatur xx. Restat $2ex+ee$, quod per postulatum primum $=2ex=2x\,dx$. Q. E. D.

x^3 different. est $=3xx\,dx$. Multipl. $x+e$, $x+e$, $x+e$. Productum est $x^3+3e\,xx+3ee\,x+e^3$. Subtr. x^3. Restat $3e\,xx+3ee\,x+e^3=$ per postul. primum $3e\,xx=3xx\,dx$. Pari ratione demonstratur, quod quant. x^4 different. $=4x^3dx$ et x^5 different. $=5x^4dx$ et x^6 different. $6x^5dx$. et similiter de caeteris.

Ex quibus sequens Regula Generalis dici potest:

Reg. 2. Quantitatis indeterminatae, quamcunque dimensionem habentis, differentialis est productum ejusdem quantitatis elevatae ad eandem dimensionem unitate diminutam, in differentialem suam, toties sumptum, quot dimensiones quantitas indeterminata habet. Sive si characteribus regulam exprimere magis juvat: x^{p} differentialis $=p\,x^{p-1}\,dx$.

[3] Quantitatis xy differentialis est $x\,dy+y\,dx$. Multipl. $x+e$ per $y+f$ (supposito $e=dx$ et $f=dy$). Prod. $xy+ey+fx+ef$ Subtrahe xy. Restat $ey+fx+ef=$ per postul. 1 $ey+fx=ydx+xdy$ q. e. d. xyz different. $=xy\,dz+zx\,dy+zy\,dx$. Multipl. $x+e$, $y+f$, $z+g$ suppon $g=dz$. Product $xyz+zye+zxf+xyg+zef+yeg+xfg+gef$. subtr. zxy restat $zye+zxf+xyg+zef+yeg+xfg+gef=$ per postulatum 1. $zye+zxf+xyg=zy\,dx+zx\,dy+xy\,dz$. Pari modo demonstratur, $xyzu$ differentialem esse $=xyz\,du+xy\,u\,dz+xz\,u\,dy+yz\,u\,dx$. Et sic de caeteris. Ex quibus etiam formari potest

Regula 3. Producti plurimarum quantitatum in se invicem differentialis est aequalis summae productorum unius cujusque differentialis in productum reliquarum.

De Quantitatum divisarum Differentialibus.

Quantitatis $\frac{1}{x}$ different. $= \frac{-dx}{xx}\cdot$ Quod sic probatur: Subtrahatur $\frac{1}{x}$ de $\frac{1+0}{x+e}\cdot$ Residuum erit $\frac{-e}{xx+ex}$ = per postul. 1 $\frac{-e}{xx}$ $= \frac{-dx}{xx}$ Q. E. D.

Vel aliter. Supponatur $\frac{1}{x} = z$ erit $1 = xz$ et sumpta differentiali utriusque membri (quia 1 determinata nullam habet differentialem) $0 = x\,dz + z\,dx$ et $dz = \frac{-z\,dx}{x} = \frac{-dx}{xx}\cdot$ Q. e. d.

$\frac{xx}{a}$ different. $= \frac{2x\,dx}{a}\cdot$ Demonstratio similis est praecedenti

$\frac{x}{y}$ different. $\frac{y\,dx - x\,dy}{yy}\cdot$ Subtr. $\frac{x}{y}$ de $\frac{x+e}{y+f}\cdot$ Restat $\frac{ey - fx}{yy+fy}$ = per postul. 1 $\frac{ey-fx}{yy} = \frac{y\,dx - x\,dy}{yy}\cdot$ Q. E. D. Aliter supp. $\frac{x}{y} = z$ erit $x = yz$ et $dx = y\,dz + z\,dy = y\,dz + \frac{x}{y}dy$ et $dx - \frac{x}{y}dy$ $= y\,dz.$ et $\frac{y\,dx - x\,dy}{yy} = dz.$ Q. e. d.

[4] Ex his iterum Regula formatur:

Reg. 4. Differentialis cujusque fractionis est productum Denominatoris in differentialem Numeratoris, minus producto Numeratoris in differentialem Denominatoris, divisum per quadratum Denominatoris.

ut si $\frac{x}{a+x}$ different. $= \frac{a\,dx}{aa+2ax+xx}\cdot$ Sic quantitatis $\frac{xy+yz}{u+t}$

differentialis est $= \dfrac{\begin{array}{l} +ux\,dy + uy\,dx - xy\,du \\ +uz\,dy + ty\,dx - yz\,du \\ +tx\,dy + uy\,dz - xy\,dt \\ +tz\,dy + ty\,dz - yz\,dt \end{array}}{uu + 2ut + tt}$

et $\frac{x-y}{u-t}$ different.

$$= \frac{+u\,dx - t\,dx - u\,dy + t\,dy - x\,du + y\,du + x\,dt - y\,dt}{uu - 2ut + xx}$$

De Quantitatum Surdarum Differentialibus.

Quantitatum sub Signo aliquo radicali contentarum differentiales sic inveniuntur: Sit e. gr. data quantitas $\sqrt{ax+xx}$, quam voco $=z$ erit $ax+xx=zz$ et $a\,dx+2x\,dx=2z\,dz=2\,dz\sqrt{ax+xx}$. Ergo $\frac{a\,dx+2x\,dx}{2\sqrt{ax+xx}}=dz=$ differentiali ipsius $\sqrt{ax+xx}$.

Eodem modo invenitur differentialis ipsius $\sqrt[3]{ax+xx}$, quae est $\frac{a\,dx+2x\,dx}{3\square\sqrt[3]{ax+xx}}$.*) Pariter quoque de $\sqrt[4]{yx+xx}$ differentialis est $=\frac{y\,dx+x\,dy+2x\,dx}{4\,C\sqrt[4]{yx+xx}}$**). Sic de $\sqrt[5]{ayx+x^3+zyx}$ different. habetur

$$\frac{ay\,dx+3xx\,dx+yz\,dx+ax\,dy+zx\,dy+xy\,dz}{5\,QQ\sqrt[5]{ayx+x^3+zyx}}\text{***)}.$$

[5] Eaedem Differentiales inveniuntur alio modo ex generatione seriei, ubi quantitates ipsae sunt in Proportione geometrica, et Potestates in Proportione Arithmetica.

$x^{(4}$	Ex. gr. Ad inveniendam differentialem ipsius $\sqrt{ax+xx}$
$x^{(3}$	considero quantitatem $ax+xx$ ut x elevatam ad potestatem $\frac{1}{2}$
$x^{(2}$	
$x^{(1}$	quae est media proportionalis inter $x^{(1}$ et $x^{(0}=1$. Et
$x^{(0}=1$	quaero per Regulam 2 dam ejus differentialem, quae est
$x^{(-1}=\frac{1}{x}$	$\frac{1}{2}$, $\overline{ax+xx}^{(-\frac{1}{2}}$, $ax+2xdx=\frac{adx+2xdx}{2\sqrt{ax+xx}}$****). Est enim x
$x^{(-2}=\frac{1}{xx}$	elevata ad potestatem $-\frac{1}{2}$, quae est media proportionalis
$x^{(-3}=\frac{1}{x^3}$	inter $x^{(0}=1$ et $x^{(-1}=\frac{1}{x}$. Idcirco $x^{(-\frac{1}{2}}=\frac{1}{\sqrt{x}}$.
$x^{(-4}=\frac{1}{x^4}$	Ergo pariter $\overline{ax+xx}^{(-\frac{1}{2}}=\frac{1}{\sqrt{ax+xx}}$. Cujus dimidium

*) In moderner Form $\frac{(a+2x)\,dx}{3\sqrt[3]{(ax+x^2)^2}}$

**) $\frac{(y+2x)\,dx+x\,dy}{4\sqrt[4]{(yx+x^2)^3}}$

***) QQ bedeutet die vierte Potenz der Wurzel.

****) Hier wie auch im folgenden vertritt das Komma das jetzige Multiplikationszeichen.

multiplicatum per $a\,dx + 2x\,dx$ facit $\frac{a\,dx + 2x\,dx}{2\sqrt{ax + xx}}$. Quod est differentiale ipsius $\sqrt{ax + xx}$. Eodem modo pro invenienda differentiali ipsius $\sqrt[3]{ax + yy}$, considero illam ut $\sqrt[3]{x} = x^{(\frac{1}{3}}$ quae est prima duarum mediarum proportionalium inter $x^{(0} = 1$ et $x^{(1}$, et invenio differentialem ejus per Reg. 2

$$\tfrac{1}{3},\ \overline{ax + yy}^{(-\frac{2}{3}},\ a\,dx + 2y\,dy = \frac{a\,dx + 2y\,dy}{3\,\square\,\sqrt[3]{ax + yy}}.$$

Adhuc aliter invenitur differentialis quantitatum surdarum per Analogiam caeterarum. Ex. gr. $\sqrt[3]{x^3}$ different. est dx, quae potest invenire dividendo differentialem $3xx\,dx$ ipsius Cubi x^3, per $3xx$, triplum $\square$ ti radicis. Item $\sqrt{xx}$ differ. $= dx$, quae potest inveniri dividendo differentialem $2x\,dx$ Quadrati xx per $2x$ duplum radicis.

[6] $\sqrt[4]{x^4}$ different. est $= dx$, quae invenitur dividendo differentialem $4x^3dx$ per $4x^3$, quadruplum Cūbi Radicis. Sic de $\sqrt[5]{x^5}$ differ. $= dx$, quae habet dividendo different. $5x^4dx$ per $5x^4$ quintuplum Quadratoquadrati Radicis. Et sic de caeteris.

Quo itaque regularum loco haberi possunt pro inveniendis differentialibus surdarum. Sic e. gr. possum etiam invenire differentialem ipsius $\sqrt[5]{x}$, dividendo nimirum differentialem dx ipsius x per quintuplum quadrato-quadrati ipsius $\sqrt[5]{x}$, quae erit $\frac{dx}{5\sqrt[5]{x^4}}$.

Eodem modo $\sqrt[4]{x}$ different. $= \frac{dx}{4\sqrt[4]{x^3}}$. et $\sqrt[3]{x}$ differentialis $= \frac{dx}{3\sqrt[3]{xx}}$ et $\sqrt{x}$ different. $= \frac{dx}{2\sqrt{x}}$.

Ex praemissis sequens Regula deduci potest ad cujuscunque quantitatis surdae differentialem inveniendam, videlicet

Reg. 5. Per eandem quantitatem surdam elevatam ad eundem dimensionis numerum unitate diminutum, qui sub signo radicali continetur, et toties sumptam quoties idem Numerus dimensionis sub signo radicali contentus, divide differentialem ejus absolutae, quotiens erit differentialis quaesita.

Regula characteribus ita exprimi potest:

$$\sqrt[p]{x}\ \text{differ.} = \frac{dx}{p\sqrt[p]{x^{p-1}}}.$$

Ex. gr. $\sqrt{ax+xx}$ differentialis est

$$= \frac{a\,dx + 2x\,dx}{2\sqrt{ax+xx}} \quad \sqrt[3]{ax+xx} \text{ different. } \frac{a\,dx+2x\,dx}{3\,\square\sqrt[3]{ax+xx}}.$$

Ad inveniendam differentialem quantitatis $\frac{\sqrt[3]{ax+xx}}{\sqrt{yx+yy}}$. Inventa differentiali $\frac{a\,dx+2x\,dx}{3\square\sqrt[3]{ax+xx}}$ Numeratoris, ut et differentiali Denominatoris [7] $= \frac{y\,dx+x\,dy+2y\,dy}{2\sqrt{yx+yy}}$ per Regulam 5. Invenitur per Regulam 4:

$$\frac{\frac{a\,dx+2x\,dx}{3\square\sqrt[3]{ax+xx}} \text{ in } \sqrt{xy+yy} + \frac{-y\,dx - x\,dy - 2y\,dy}{2\sqrt{xy+yy}} \text{ in } \sqrt[3]{ax+xx}}{xy+yy}$$

differentialis ipsius $\frac{\sqrt[3]{ax+xx}}{\sqrt{yx+yy}}$. Quantitatis $\sqrt{ax+xx+\sqrt{aay+y^3}}$ differentialis est $= \frac{2a\,dx+4x\,dx \text{ in } \sqrt{aay+y^3} + aa\,dy + 3yy\,dy}{2\sqrt{ax+xx+\sqrt{aay+y^3}} \text{ in } 2\sqrt{aay+y^3}}$.

[8] Diese Seite ist leer.

[9] De Usu Calculi Differentialis in resolvendis Problematibus.

Problema I.

Invenire Tangentem Parabolae.

Est ex natura Parabolae $ax = yy$. Ergo etiam $a\,dx = 2y\,dy$. Igitur $a \cdot 2y :: dy \cdot dx$.*) porro, quia unaquaevis linea Curva ex infinitis lineis rectis constare intelligitur per Postul. 2 erit Tangens AD et portio infinite parva DF parabolae BDF una linea recta: Ducta itaque (Fig. 1) Diametro AE parallela DG, erunt $\triangle$la DGF et ACD similia. Quocirca $FG \cdot GD :: CD \cdot AC$ id est $dy \cdot dx :: y \cdot s$ (subtangentem) :: a $\cdot\, 2y$ (ex anteced.) igitur $s = \frac{2yy}{a} = \frac{2ax}{a} = 2x$. Si itaque AC sumatur dupla ipsius abscissae BC, et per punctum A et datum in Curva punctum D ducatur recta AD, erit illa Tangens. Q. e. i.

*) In heutiger Schreibweise $a : 2y = dy : dx$.

Eodem modo in Parabola Cubicali invenitur Tangens:

1. Si ejus natura sit $aa\,x = y^3$. Subtangens erit $= 3x$. Est enim $aa\,dx = 3yy\,dy$, et $aa \cdot 3yy :: dy \cdot dx :: y \cdot s$

$$= \frac{3y^3}{aa} = \frac{3aa\,x}{aa} = 3x\ \text{*)}.$$

2. Si $a\,xx = y^3$. Erit $s = \frac{3x}{a}$. Nam $2a\,dx = 3yy\,dy$. Quapropter $2ax \cdot 3yy :: dy \cdot dx : : y \cdot s = \frac{3y^3}{2ax} = \frac{3a\,xx}{2ax} = \frac{3x}{2}$.

In Parabola Biquadratica.

1. Si $a^3 x = y^4$. $s = 4x$. Est enim $a^3 dx = 4y^3 dy$. Ergo $a^3 \cdot 4y^3 :: dy \cdot dx :: y \cdot s = \frac{4y^4}{a^3} = \frac{4a^3x}{a^3} = 4x$.

[10] 2. Si $aa\,xx = y^4$. $s = 2x$. Est enim $2aax\,dx = 4y^3 dy$ et $2aax \cdot 4y^3 :: dy \cdot dx :: y \cdot s = \frac{4y^4}{2aax} = \frac{4aa\,xx}{2aax} = 2x$.

3. Si $ax^3 = y^4$. $s = \frac{4}{3}x$. $3axx\,dx = 4y^3 dy$ et $3axx \cdot 4y^3 :: dy \cdot dx :: y \cdot s = \frac{4y^4}{3axx} = \frac{4ax^3}{3axx} = \frac{4x}{3}$.

In Parabolis Caeteris.

1. Si $a^4 x = y^5$. $s = 5x$. $a^4 dx = 5y^4 dy$
2. Si $a^3 xx = y^5$. $s = \frac{5}{2}x$
3. Si $aax^3 = y^5$. $s = \frac{5}{3}x$
4. Si $ax^4 = y^5$. $s = \frac{5}{4}x$

Et sic in Caeteris. Ex quibus Regula generalis formari potest: Si Parabolae cujuscunque natura sit $a^{(c}\, x^{(m} = y^{(n}$.

Ejus Subtangens erit $\frac{nx}{m}$.

Est enim $a^c\, mx^{m-1} dx = ny^{n-1} dy$, et $a^c\, mx^{m-1} \cdot ny^{n-1}$

$$:: dy \cdot dx : y \cdot s = \frac{ny^n}{a^c\, mx^{m-1}} = \frac{a^c\, x^m\, n}{a^c\, x^{m-1}\, m} = \frac{a^c\, x^{m-1}\, nx}{a^c\, x^{m-1}\, m} = \frac{nx}{m}.$$

*) Die erste linke Seite der fortlaufenden Gleichung ist natürlich nur s; wie auch bei den folgenden entsprechenden Gleichungen.

[11] *Problema II.*

Invenire Tangentem Ellipseos.

Sit (Fig. 2) Diameter $BJ = b$. Parameter $= a$ Abscissa $= x$ ordinatim applicata $= y$. CE abscissae differentialis $= dx$, et FG ordinatim applicatae differ. $= dy$. Sunt hic (eadem ratione, ut in Parabola notavimus) $\triangle$la DGF et ACD similia. Idcirco $FG \cdot GD :: DC \cdot AC$. i. e. $dy \cdot dx :: y \cdot s$. Porro ex natura Ellipseos $b \cdot a :: bx - xx \cdot yy$. Ergo $abx - axx = byy$, et sumptis utrinque differentialibus $ab\,dx - 2ax\,dx = 2by\,dy$. Igitur $ab - 2ax \cdot 2by :: dy \cdot dx :: y \cdot s$. Est itaque

$$s = \frac{2byy}{ab - 2ax} = \frac{2abx - 2axx}{ab - 2ax} = \frac{2bx - 2xx}{b - 2x}.$$

Problema III.

Invenire Tangentem Hyperbolae.

Iisdem positis quae in Ellipsi, est etiam $dy \cdot dx :: y \cdot s$. Et (Fig. 3) ex natura Hyperbolae $b \cdot a :: bx + xx \cdot yy$. Idcirco $abx + axx = byy$ et $abdx + 2axdx = 2bydy$. igitur $ab + 2ax \cdot 2by :: dy \cdot dx :: y \cdot s =$

$$= \frac{2byy}{ab + 2ax} = \frac{2ab\,x + 2a\,xx}{ab + 2ax} = \frac{2bx + 2xx}{b + 2x}.\text{*)}$$

Inventa Tangente inveniuntur etiam Asymptotae, considerando eas ut Tangentes [12] in infinito, earumque abscissas x et applicatas y, ut infinitas. Et inveniendo JH et JM, per quarum terminos H et M Asymptota transit. Est (Fig. 4) autem JH hic eadem quae in caeteris Tangentibus est AJ, et JM eadem quae JO, videlicet $JH = \frac{bx}{b + 2x} =$ per postul. 1 (quoniam x hic est infinite major quam b) $\frac{bx}{2x} = \frac{1}{2}\,b = \frac{1}{2}$ Diametro transversae; et $JM = \frac{by}{2b + 2x}$ $= \frac{\sqrt{abx}}{\sqrt{4b + 4x}} =$ per postul. 1 $\sqrt{\frac{abx}{4x}} = \frac{\sqrt{ab}}{2} =$ semidiametro Conjugatae.

Problema IV.

Invenire Tangentem Curvae, quae habet hanc proprietatem, ut summa trium linearum rectarum a quovis puncto in curva, ad tria puncta data in aliqua recta, ductarum, semper sit aequalis.

*) Siehe vorige Fussnote.

[13] Sint (Fig. 5) $A.B.C.$ tria puncta data, eorumque distantiae $AB = a$ $AC = b$ $AF = x$ $DF = y$ Summa trium linearum $AD + BD + CD = c$. Erit igitur $AD = \sqrt{xx + yy}$ $BD = \sqrt{aa - 2ax + xx + yy}$ et $DC = \sqrt{bb - 2bx + xx + yy}$. Ergo

$$\sqrt{xx + yy} + \sqrt{aa - 2ax + xx + yy} + \sqrt{bb - 2bx + xx + yy} = c$$

et sumptis utrinque differentialibus

$$\frac{2x\,dx + 2y\,dy}{2\sqrt{xx + yy}} + \frac{2x\,dx + 2y\,dy - 2a\,dx}{2\sqrt{aa - 2ax + xx + yy}} + \frac{2x\,dx + 2y\,dy - 2b\,dx}{2\sqrt{bb - 2bx + xx + yy}} = 0.$$

Reductis quantitatibus, in quibus est dx ad unam partem, et in quibus est dy ad alteram, divisaque tota quantitate per $\frac{2}{2}$ erit

$$\frac{y\,dy}{\sqrt{xx + yy}} + \frac{y\,dy}{\sqrt{aa - 2ax + xx + yy}} + \frac{y\,dy}{\sqrt{bb - 2bx + xx + yy}}$$
$$= \frac{b\,dx - x\,dx}{\sqrt{bb - 2bx + xx + yy}} + \frac{a\,dx - x\,dx}{\sqrt{aa - 2ax + xx + yy}} + \frac{-x\,dx}{\sqrt{xx + yy}}.$$

Resoluta aequatione in proportionem, erit

$$\frac{y}{\sqrt{xx + yy}} + \frac{y}{\sqrt{aa - 2ax + xx + yy}} + \frac{y}{\sqrt{bb - 2bx + xx + yy}}$$
$$\cdot \frac{b - x}{\sqrt{bb - 2bx + xx + yy}} + \frac{a - x}{\sqrt{aa - 2ax + xx + yy}} + \frac{-x}{\sqrt{xx + yy}}$$
$$:: dx \cdot dy$$

et $dy \cdot dx = y \cdot s$ (propter triangula similia DDG et EDF) quae igitur

$$\frac{\dfrac{yy}{\sqrt{xx + yy}} + \dfrac{yy}{\sqrt{aa - 2ax + xx + yy}} + \dfrac{yy}{\sqrt{bb - 2bx + xx + yy}}}{\dfrac{b - x}{\sqrt{bb - 2bx + xx + yy}} + \dfrac{a - x}{\sqrt{aa - 2ax + xx + yy}} - \dfrac{x}{\sqrt{xx + yy}}}.$$

Ex hac resolutione manifeste apparet, hanc methodum breviorum esse, et magis succinctam, quam Cartesii, per quam, si hoc Problema resolvendum institueretur, oporteret primo, cujus generis sit haec linea Curva, invenire, ut et aequationem ex puris rationalibus constantem; quod opus est magni laboris et taedii.

[14] *Problema V.*

Invenire qualis linea sit ea, cujus Subtangentes semper sunt aequales.

Sit (Fig. 6) AB Tangens, BC Subtangens $= a$, $AC = y$, $AD = dy$, cC vel $aD = dx$. Erit $dy \cdot dx :: y \cdot a$. et alternando $dy \cdot y :: dx \cdot a$.

Quia vero ratio $dx \cdot a$ semper est constans, erit etiam $dy \cdot y$ semper constans i. e. $\div y \cdot y \cdot y$ faciunt progressionem Geometricam. Estque ideo haec linea Logarithmica, cujus ordinatim applicatae faciunt progressionem Geometricam et abscissae Arithmeticam.

Problema VI.

Invenire Tangentem Cycloidis.

Sit (Fig. 7) ABC Cyclois, cujus Tangens invenienda sit in puncto E, ducatur in medio hujus circulus $BHDB$, cujus semicircumferantia aequalis dimidiae basi AD, vel DC, et diameter $= 2a$. Ducatur porro EM parallela ipsi basi AC, eidemque parallela $BF = x$ et ad hanc ordinata $EF = y = BM$. Estque per naturam Curvae recta $EH =$ arcui $HB = f$. adeoque $x = FB = EH + HM =$

$$f + \sqrt{2ay - yy} \quad \text{et} \quad dx = df + \frac{2a\,dy - 2y\,dy}{2\sqrt{2ay - yy}}.$$

Est autem [15] $df = HN =$ per postul. 2. subtensae trianguli Rectanguli $HKN = \sqrt{\Box HK + \Box KN} = \dfrac{a\,dy}{\sqrt{2ay - yy}}$. Igitur

$$dx = \frac{2a\,dy - y\,dy}{\sqrt{2ay - yy}} \quad \text{et quia } dy \cdot dx :: y \cdot s \text{ erit etiam}$$

$$dy \cdot \frac{2a\,dy - y\,dy}{\sqrt{2ay - yy}} :: y \cdot s \text{ id est } 1 \cdot \frac{2a - y}{\sqrt{2ay - yy}} :: y \cdot s \text{ quae}$$

$$\text{ergo} = \frac{2ay - yy}{\sqrt{2ay - yy}} = \sqrt{2ay - yy} = HM. \text{ Quia subtangens}$$

$FG = HM$ erit $FB - FG = EM - HM$ i. e. $GB = EH$ et per consequens Tangens EG aequalis et parallela subtensae BH.

Problema VII.

Invenire Tangentem Conchoidis.

Sit (Fig. 8) $GL = a$
$CF = AD = b$
$GD = x$
$DE = AB = dx$.

1.° Propter triangula similia DEF et DLG. $DL \cdot LG :: DE \cdot EF$

$$\sqrt{xx - aa} \cdot a :: dx \cdot \frac{a\,dx}{\sqrt{xx - aa}}.$$

2.° Propter triangula similia BGC et EGF. $GF \cdot GC :: EF \cdot BC$

$$x \cdot b + x :: \frac{a\,dx}{\sqrt{xx - aa}} \cdot \frac{ax\,dx + ab\,dx}{x\sqrt{xx - aa}}.$$

3.° Propter triangula similia ABC et AGK. $AB \cdot BC ::$ $AG \cdot GK$ i. e. $dx \cdot \frac{ax\,dx + ab\,dx}{x\sqrt{xx - aa}} :: x + b \cdot GK$, seu diviso primo et secundo termino per dx.

$$1 \cdot \frac{ax + ab}{x\sqrt{xx - aa}} :: x + b \cdot \frac{axx + 2abx + abb}{x\sqrt{xx - aa}} = GK.\text{*)}$$

[16] Absque calculo sic resolvitur:
$AB \cdot BC :: ED \cdot BC$ (quia $ED = AB$) $:: ED \cdot EF + EF \cdot BC ::$ $DG \cdot GH + GF \cdot GC.$ $:: \square DGF \cdot \square CGH :: AG \cdot GK.$**)
vel $\square DG$.

Constructio.

Tangens ex his facile construitur hoc modo: Ducatur CM parallela ipsi FH connexisque punctis F et M, fiat AK parallela connectenti FM, erit haec AK Tangens quaesita.

Demonstratio.

Quia $CG \cdot GM :: FG \cdot GH$. erit $\square CGH = \square FGM$. Sed $AG \cdot GK :: FG \cdot GM :: \square FG \cdot \square FGM$ ($\square CGH$). Ergo $AG \cdot GK$ $:: \square FG$ vel $\square DG \cdot \square CGH$. ideoque per calculum inventa erit AK Tangens Conchoideos.

Problema VIII.

Determinare Tangentem in Curva Cissoide.

Sit (Fig. 9) ABC semicirculus, FB perpendicularis ex Centro elevata, BD, BE sunt arcus aequales quomodocunque sumpti, intersectis H, linea ducta AE et perpendicularis DG, est punctum in Cissoide. Oportet nunc determinare Tangentem in hoc puncto. In hunc finem quaeratur aequatio naturam Curvae exprimens, quod sic fit: Sit $AF = FC = a$ $AG = x$ $GH = y$. Ergo FG vel $FK = a - x$ et GD vel $KE = \sqrt{2ax - xx}$. Est autem $AK \cdot KE :: AG \cdot GH$ i. e. $2a - x \cdot \sqrt{2ax - xx} :: x \cdot y$ ideoque $\sqrt{2a - x} \cdot \sqrt{x} :: x \cdot y$ vel $2a - x \cdot x :: xx \cdot yy$. Proinde $x^3 = 2ayy - xyy$. eorumque diffe-

*) Also $GK = \frac{ax^2 + 2abx + ab^2}{x\sqrt{x^2 - a^2}}$

**) Diese letzten Proportionen würde man jetzt etwa schreiben: da $ED = AB$ ist, so ist:

$$AG : GK = AB : BC = ED : BC = \frac{ED}{EF} \cdot \frac{EF}{BC} = \frac{DG}{GH} \cdot \frac{GF}{GC}.$$

Nun ist $GF = GD$, also $DG^2 : (GH \cdot GC) = AG : GK$.

rentialis [17] $3xxdx = 4aydy - 2xydy - yydx$ et $3xxdx + yydx = 4aydy - 2xydy$. Ideoque $3xx + yy \cdot 4ay - 2xy :: dy \cdot dx :: y \cdot s$ erit ergo s vel

$$GL = \frac{4ayy - 2xyy}{3xx + yy} = \frac{2x^3}{3xx + yy}$$

vel substituto valore ipsius yy, provenit $\frac{2ax - xx}{3a - x}\cdot$

Problema IX.

Invenire Tangentem in Quadratrice.

Si (Fig. 10) ABC est Quadrans Circuli, fiatque quilibet arcus AD, ad portionem radii AE, ut quadrans AB ad totum radium AC, erit, ductis radio DC et perpendiculari EF, punctum intersectionis F in Curva AFG, quae vocatur Quadratrix: petitur jam, ut in puncto F determinatur Tangens. Sit $AC = a$ $AB = b$ $AH = x$ $AD = f$ erit $DH = \sqrt{2ax - xx}$ $AE = \frac{af}{b}$ $HC = a - x$ $EC = a - \frac{af}{b}\cdot$ Est autem $HC \cdot HD :: EC \cdot EF$. id est $a - x \cdot \sqrt{2ax - xx} :: \frac{ab - af}{b} \cdot EF$.

Invenitur ergo $EF = \frac{\overline{ab - af}\,\sqrt{2ax - xx}}{ab - bx}\cdot$ Nunc, ut in Problemate Sexto, reperitur pro portiuncula Dd i. e. pro $df = \frac{adx}{\sqrt{2ax - xx}}$, ideoque pro portiuncula Ee, id est pro differentiali lineae $AE = \frac{aadx}{b\sqrt{2ax - xx}}$, et sumpta per Reg. 4 et 5 differentiali ipsius EF (ubi substituatur valor ipsius df) faciendum est, ut differentialis lineae EF ad differentialem lineae EA, ita linea EF ad s.

Aliter.

Sit (Fig. 11) CK perpendicularis DC, et retentis iisdem literis, quas prius, quaeratur punctum K, quod si conjungatur cum puncto F, linea FK tangat Curvam AFG, id fit hoc modo: $HC \cdot DC :: EC \cdot CF$ i. e. $a - x \cdot a :: \frac{ab - ef}{b} \cdot CF$.

[18] Erit ergo $FC = \frac{aab - aaf}{ab - bx}\cdot$ Item $DC \cdot FC :: Dd \cdot Ff$ id est $a \cdot \frac{aab - aaf}{ab - bx} :: df = \frac{adx}{\sqrt{2ax - xx}} \cdot Ff$ ideoque erit haec Ff

aequalis $\frac{aab\,dx - aaf\,dx}{\overline{ab - bx}\sqrt{2ax - xx}}$. Sed differentialis ipsius $FC =$

$$\frac{-a^3b\,df + aaxb\,df + aabb\,dx - aafbdx}{aabb - 2abbx + bbxx}$$

$$= \frac{-a^4b\,dx + a^3bx\,dx + \overline{aabb\,dx - aafb\,dx}\sqrt{2ax - xx}}{\overline{aabb - 2abbx + bbxx}\sqrt{2ax - xx}}$$

$=$ differ. ipsius FC. Sed haec differentialis se habet ad

$$\frac{aab\,dx - aaf\,dx}{\overline{ab - bx}\sqrt{2ax - xx}} \text{ ut } \frac{-aa + ax + \overline{b - f}\sqrt{2ax - xx}}{a - x} \cdot b - f$$

$$:: \frac{aab - aaf}{ab - bx} = FC \cdot CK = \frac{aabb - 2aabf + aaff}{-aab + abx + \overline{bb - bf}\sqrt{2ax - xx}} \text{*)}.$$

Si itaque Tangens in puncto A ducenda fit, erit $CK = -b$. quae quantitas, quia est negativa, ostendit, quod CK ad dextram i. e. ad partem ipsi AH vel x adversam, sit sumenda.

Problema X.

Invenire punctum intersectionis G (Fig. 12) curvae Quadratricis AG et radii perpendicularis CB.

Intelligatur punctum D adeo jam approximasse puncto B, ut distantia DB sit infinite parva, sicut et [19] distantia CE, ductis itaque radio CD et perpendiculari EF, erit haud dubie punctum F idem censendum quod G, quippe quae ab invicem non distant, nisi intervallo infinite parvo. Hoc autem punctum F ita determinatur: $AB \cdot AC :: AD \cdot AE :: DB \cdot EC :: DB \cdot FG :: CB \cdot CG :: AC \cdot CG$. est itaque CG tertia proportionalis ad quadrantem peripheriae et ad radium. Hinc punctum G aliter determinari nequit, quin simul rectificatio lineae Circularis habeatur.

Problema XI **).

Invenire Tangentem in Spirali Archimedea.

Spiralis Archimedea vocatur illa Curva, quae describitur a puncto, quod a Centro ad peripheriam Circuli movetur, in eodem radio aequabiliter et aequali temporis spatio, quo punctum a Centro ad Peripheriam movetur, circumrotante. Incumbit nunc hujus

---

*) Also $\frac{-a^2 + ax + (b-f)\sqrt{2ax - x^2}}{a - x} : b - f = FC : CK$.

Da nun $FC = \frac{a^2b - a^2f}{ab - bx}$ ist, so ist $CK = \frac{a^2b^2 - 2a^2bf + a^2f^2}{-a^2b + abx + (b^2 - bf)\sqrt{2ax - x^2}}$

**) In der Handschrift steht irrtümlich IX.

Curvae Tangentem invenire. Sit (Fig. 13) radius $AC = a$ peripheria $DDCD = b$ $AB = x$, et ducatur perpendicularis AE ad lineam AB. Erit Radius AC ad peripheriam ut AB ad arcum CKD. Item $AD \cdot AF :: DD \cdot FG$ $a \cdot x :: \frac{b\,dx}{a} \cdot \frac{bx\,dx}{aa}$ porro $BG \cdot FG :: AB \cdot AE$ i. e. $dx \cdot \frac{bx\,dx}{aa} :: x \cdot \frac{bxx}{aa} = s$ *). Sic si Tangens in puncto C ducenda sit, invenitur $s = b$ quod Archimedes longo discursu demonstravit.

[20] *Problema XII.*

De Maximis et Minimis.

Ad inveniendam quantitatem maximam, considerantur quantitates, ut ordinatim applicatae alicujus curvae concavae versus axem, ut in Fig. ABC (Fig. 14). Et vice versa ad inveniendam quantitatem minimam, considerantur ut applicatae alicujus Curvae convexae versus axem, ut in Fig. GDE, ubi GE axis. His consideratis ducatur Tangens in puncto maximae vel minimae, quae erit parallela Axi. Quia $dy \cdot dx :: y \cdot s$ et y infinities minor quam Subtangens, erit etiam $dy = 0$ respectu ipsius dx. Ita si inveniendum sit maximum rectangulum eorum, quae faciunt duae partes x et $a - x$ lineae datae a, considero $ax - xx$ ut applicatam in quadam Curva concava versus axem, et ejus differentialem $a\,dx - 2x\,dx = 0$. Idcirco $a\,dx = 2x\,dx$ et $x = \frac{1}{2}a$. Erit igitur maximum rectangulum, si x sumatur $= \frac{1}{2}a$.

Problema XIII.

Dividere lineam datam in tres partes, ita ut partes multiplicatae invicem faciant maximum solidum, quod potest produci a tribus partibus ejusdem lineae.

Sit (Fig. 15) pars $AB = x$, erit reliqua bisecanda in puncto D est autem per Probl. XII $\square BD$ sive $\square DC$ maximum rectangulum duarum partium lineae BC, ergo idem multiplicatum per partem AB etiam maximum [21] solidum trium partium ejusdem lineae. Sic $aax - 2axx + x^3 =$ Maximo. Ejusque differentiale $aa\,dx - 4ax\,dx + 3xx\,dx = 0$ et $xx = \frac{4}{3}ax - \frac{1}{3}aa$ et $x = \frac{1}{3}a$.

Eodem modo, si linea AC secanda sit in quatuor partes, ita ut partes multiplicatae in se invicem faciant maximam quanti-

*) Anders ausgedrückt: aus der ersten Proportion folgt $FG = \frac{b\,x\,dx}{a^2}$ und demnach aus der zweiten $AE = s = \frac{b\,x^2}{a^2}$.

tatem quatuor dimensionum, invenitur $x = \frac{1}{4}a$. Et si linea AC secanda sit in quinque partes, etc. invenitur $x = \frac{1}{5}a$. Et sic de caeteris.

Problema XIV.

Invenire maximum Rectangulum eorum, quae describuntur ab abscissis et ordinatim applicatis in Circulo.

Sit (Fig. 16) Diameter $AB = a$ et $AC = x$ erit $CB = a - x$ et $CD = \sqrt{ax - xx}$ ▭ $ACD = \sqrt{ax^3 - x^4} =$ Maximo. Ejusque differentiale

$$\frac{3axx\,dx - 4x^3dx}{2\sqrt{ax^3 - x^4}} = 0.$$

Et $3axx\,dx = 4x^3dx$ et $3a = 4x$ et denique $x = \frac{3}{4}a$. Q. e. i.

Problema XV.

Invenire maximum Rectangulum eorum, quae describuntur ex portionibus ordinatim applicatarum in quadrante Circuli sectarum per subtensam ipsius Quadrantis.

Sit (Fig. 17) $AC = a$ $DC = x$ erit $DF = x$ et $DE = \sqrt{2ax - xx}$ erit ▭ $DFE = \sqrt{2ax^3 - x^4} - xx =$ Maximo. Ejusque differentiale

$$\frac{6axx\,dx - 4x^3dx}{2\sqrt{2ax^3 - x^4}} - 2x\,dx = \frac{3ax\,dx - 2xx\,dx - 2x\,dx\sqrt{\cdots}}{\sqrt{2ax^3 - x^4}} = 0.$$

Ergo $3a - 2x = 2\sqrt{2ax^3 - x^4}$ et $9aa - 12ax + 4xx = 8ax - 4xx$ et $xx = \dfrac{20ax - 9aa}{8}$ et $x = \frac{5}{4}a \between \sqrt{\frac{7}{16}aa}$ *).

[22] *Problema XVI.*

Viator A (Fig. 18) tendens in E pertransire debet campum planum et tritum $AFDB$ et locum asperum et inaequalem $DBGE$, quae viae ita se habent, ut tempore a absolvatur spatium b in campo plano FDB, et eodem tempore spatium c in loco aspero DBG. Quaeritur brevissima via de A ad E i. e. quam Viator absolvit in brevissimo temporis spatio. Ducantur ad lineam BD, quae duas diversas vias dividit, perpendiculares $AB = m$ et $ED = n$. Sitque $BC = x$ $BD = e$ erit $DC = e - x$ et $AC = \sqrt{mm + xx}$ et $CE = \sqrt{ee - 2ex + xx + nn}$. Sed $b \cdot a :: \sqrt{mm + xx} \cdot \dfrac{a\sqrt{mm + xx}}{b}$ = tempus quo absolvitur linea AC et

*) $\between$ bedeutet + oder −.

$$c \cdot a :: \sqrt{ee - 2ex + xx + nn} \cdot \frac{a\sqrt{ee - 2ex + xx + nn}}{c}$$

= tempus quo absolvitur linea CE. Est igitur

$$\frac{a\sqrt{mm + xx}}{b} + \frac{a\sqrt{ee - 2ex + xx + nn}}{c}$$

= minimo temporis spatio. Ejusque differ.

$$\frac{ax\,dx}{b\sqrt{mm + xx}} + \frac{ax\,dx - ae\,dx}{c\sqrt{ee - 2ex + xx + nn}} = 0. \text{ Igitur}$$

$$\frac{x}{b\sqrt{mm + xx}} = \frac{e - x}{c\sqrt{ee - 2ex + xx + nn}}$$

et $cc\,ee\,xx - 2cc\,ex^3 + cc\,x^4 + cc\,nn\,xx =$

$= bb\,ee\,mm + bb\,ee\,xx - 2bbe\,mm\,x - 2bbe\,x^3 + bb\,mm\,xx + bb\,x^4.$

$$\text{Et} \begin{matrix} +bb \\ -cc \end{matrix} x^4 \begin{matrix} -2bbe \\ +2cce \end{matrix} x^3 \begin{matrix} +bb\,mm \\ +bb\,ee \\ -cc\,ee \\ -cc\,nn \end{matrix} xx - 2bbe\,mm\,x + bb\,ee\,mm = 0.$$

[23] *) *Problema XVII.*

In linea CE (Fig. 19) invenire punctum D, a quo si ducantur ad puncta data A et B, lineae DA, DB, ut summa earum sit minima omnium duarum linearum a punctis A et B ad punctum quendam lineae CE ductarum.

Demittantur perpendiculares $AC = a$ $BE = b$. Sitque $CE = c$ et $CD = x$. Erit $DE = c - x$.

$$AD = \sqrt{aa + xx} \text{ et } BD = \sqrt{cc - 2cx + xx + bb}.$$

Et $\sqrt{aa + xx} + \sqrt{cc - 2cx + xx + bb}$ = minimae. Ejusque differentialis

$$\frac{x\,dx}{\sqrt{aa + xx}} + \frac{x\,dx - c\,dx}{\sqrt{cc - 2cx + xx + bb}} = 0. \text{ Ergo}$$

$$\frac{x}{\sqrt{aa + xx}} = \frac{c - x}{\sqrt{cc - 2cx + xx + bb}}$$

$cc\,xx - 2cx^3 + x^4 + bb\,xx = aa\,cc - 2aa\,cx + aa\,xx + cc\,xx - 2cx^3 + x^4$, $bb\,xx = aa\,cc - 2aa\,cx + aa\,xx$, proinde $bx = ac - ax$ et $bx + ax = ac$. Et denique $x = \frac{ac}{a + b}$.

*) Diese Seite der Handschrift ist als Faksimile auf der letzten Figurentafel gegeben.

Problema XVIII.

In Radio AC (Fig. 20) invenire punctum D, a quo si ducatur perpendicularis DE ad AC, ut abscissa FE inter peripheriam BEC et subtensam BC contenta, sit maxima omnium, quae eodem modo in Quadrante duci possunt.

Sit $AC = a$ $DC = x$ erit et $FD = x$ et $DE = \sqrt{2ax - xx}$, $FE = \sqrt{2ax - xx} - x =$ Maximae. Ejusque differentiale

$$\frac{a\,dx - x\,dx}{\sqrt{2ax - xx}} - dx = 0.$$

Proinde erit $a - x = \sqrt{2ax - xx}$, $aa - 2ax + xx = 2ax - xx$.

$$xx = \frac{4ax - aa}{2} \text{ et } x = a \mp \sqrt{\tfrac{1}{2}aa}.$$

[24] *Problema XIX.*

Sit pondus A (Fig. 21) appensum funiculo AC fixo in puncto C et transeunte supra trochleam E, qua libere dependet in funiculo affixo in B. Quaeritur ubinam Trochlea E et pondus A quiescunt.

Supposito quod funiculi et trochlea nullam habeant gravitatem, quiescet Trochlea et pondus ibi, ubi distantia AD ponderis A a linea BC horizonti parallela est maxima. Quae igitur invenienda sit longitudo funiculi $AC = a$ $BC = b$ $BE = c$ et

$DE = x$ erit $BD = \sqrt{cc - xx}$. $DC = b - \sqrt{cc - xx}$.

$CE = \sqrt{bb + cc - 2b\sqrt{cc - xx}}$. $AE = a - \sqrt{bb + cc - 2b\sqrt{cc - xx}}$.

$AD = x + a - \sqrt{bb + cc - 2b\sqrt{cc - xx}} =$ Maximae. Eiusque different.

$$dx - \frac{bx\,dx}{\sqrt{cc - xx} \text{ in } \sqrt{bb + cc - 2b\sqrt{cc - xx}}} = 0.$$

Proinde

$$\sqrt{bb + cc - 2b\sqrt{cc - xx}} = \frac{bx}{\sqrt{cc - xx}}$$

et

$$bb + cc - 2b\sqrt{cc - xx} = \frac{bb\,xx}{cc - xx}.\quad bb + cc - \frac{bb\,xx}{cc - xx}$$

$$= 2b\sqrt{cc - xx} = \frac{bb\,cc + c^4 - 2bb\,xx - cc\,xx}{cc - xx}.$$

Ergo $b^4c^4 + c^8 + 4b^4x^4 + c^4x^4 + 2bbc^6 - 4b^4cc\,xx - 6bbc^4xx - 2c^6xx + 4bb\,cc\,x^4 = 4bb\,c^6 - 12bb\,c^4xx + 12bb\,cc\,x^4 - 4bb\,x^6$.

Aliter.

Sit $BD = x$. Erit $DC = b - x$ $DE = \sqrt{cc - xx}$ $CE = \sqrt{bb - 2bx + cc}$ $AE = a - \sqrt{bb - 2bx + cc}$ $AD = a - \sqrt{bb - 2bx + cc} + \sqrt{cc - xx} =$ Maximae. Ejusque differentialis =

$$\frac{+b\,dx}{\sqrt{bb - 2bx + cc}} - \frac{x\,dx}{\sqrt{cc - xx}} = 0.$$

Ergo $x\sqrt{bb - 2bx + cc} = b\sqrt{cc - xx}$. Et $bb\,xx - 2bx^3 + cc\,xx =$

$$bb\,cc - bb\,xx \text{ et } x^3 = \frac{2bb\,xx + cc\,xx - bb\,cc}{2b}.$$

[25] *Problema XX.*

Invenire brevissimum Crepusculum.

Sit (Fig. 22a) C centrum Sphaerae, ANB Meridianus, AB Diameter Aequatoris, MN diameter Horizontis, OP diam. paralleli crepuscularis, DF diameter paralleli Aequatoris, qui quaeritur et talis est, ut, cum Sol in eo versatur, arcus KL, qui intercipitur inter Horizontem et parallelum Crepuscularem, sit proportionaliter minimus, id est ut minimam rationem habeat ad suam peripheriam, vel ad suam Diametrum. Patet enim hoc in Casu Crepusculum omnium esse brevissimum.

Sit (Fig. 22b) nunc radius Sphaerae $CA = a$. GH vel sinus arcus crepuscularis $PN = b$. GH est ad GJ, ut sinus complementi elevationis poli ad sinum totum. Sit ergo $GJ = c$ proinde $HJ = \sqrt{cc - bb} = f$. Sit CE vel sinus declinationis solis quaesitae $BF = x$. Quia $GH \cdot HJ :: CE \cdot EJ$ erit $EJ = \frac{fx}{b} =$ sinui arcus RK, et $EG = \frac{bc - fx}{b} =$ sinui arcus RL: describatur nunc separatim (Fig. 22c) circulus major drf, cujus centrum e, diameter df, radius er perpendicularis ad Diametrum; fiat ut DE ad EJ ita de ad ei; et ut DE ad EG ita de ad eg, erit ductis parallelis ik, gl arcus kr similis kR et arcus rl [26] similis arcui RL et totus kl similis toti KL; quia autem arcus KL debet esse proportionaliter minimus, erit arcus kl ob constantem radium absolute minimum ideoque subtensa kl minima.

Hoc itaque modo ex assumta x inveniendae essent ie, eg, ex his ik, kl, ex quibus dein subtensa kl inveniri potest, quae more consueto debet aequari minimae, ut aequatio proveniat, quae x determinet; Verum quia subtensa kl valde composita invenitur, erit hic modus admodum prolixus, et aequationem dabit, pro determinatione ipsius x, quae sex dimensionum erit, et ultra triginta terminos continebit, adeo ut problema per Methodum Cartesianem solutu tantum non plane impossibile sit.

Videamus autem quo pacto res per Calculum Differentialem expediatur. Primo hic non necesse est, ut quaeratur subtensa, vel sinus, vel alia recta, quae determinet arcum kl, sicuti Cartesii Geometria id postulat, circa dimensionem enim curvarum illa non versatur; nobis sufficit, quod arcus kl debeat esse minimus, proinde ejus differentialis $= 0$, vel differentialis arcus $kr =$ different. arcus rl; Unicum igitur superest, ut quaerantur different. arcuum kr, rl, quibus adaequatis aequatio proveniet quaesita, cujus radix x erit sinus declinationis solis quae quaeritur. Differentialia autem arcuum kr, rl ita reperiuntur: quia $DE = \sqrt{aa - xx}$ et per Constructionem $DE \cdot JE :: de \cdot ie$ erit

$$ie = \frac{afx}{b\sqrt{aa - xx}},$$

eodem modo erit

$$eg = \frac{abc - afx}{b\sqrt{aa - xx}}$$

proinde

$$ik = \sqrt{\frac{a^4 bb - aa\, bb\, xx - aa\, ff\, xx}{aa\, bb - bb\, xx}} = \text{propter } (bb + ff = cc)$$

$$\sqrt{\frac{a^4 bb - aa\, cc\, xx}{aa\, bb - bb\, xx}},$$

et

$$gl = \sqrt{\frac{a^4 bb - aa\, bb\, xx - aa\, bb\, cc + 2aa\, bcfx - aa\, ff\, xx}{aa\, bb - bb\, xx}}$$

$$= (\text{propter } bb + ff = cc)$$

$$[27]\ \sqrt{\frac{a^4 bb - aa\, cc\, xx + 2aa\, bcfx - aa\, bb\, cc}{aa\, bb - bb\, xx}}.$$

Si nunc ie appellatur m erit $ik = \sqrt{aa - mm}$, et differentialis

arcus $kr = \dfrac{a\,dm}{\sqrt{aa - mm}}$, quia itaque $m = \dfrac{afx}{b\sqrt{aa - xx}}$,

erit

$$dm = \frac{a^3 f\, dx}{aab - bxx\sqrt{aa - xx}},$$

et quia

$$\sqrt{aa - mm} = \sqrt{\frac{a^4 bb - aa\, cc\, xx}{aa\, bb - bb\, xx}},$$

erit $\dfrac{a\,dm}{\sqrt{aa-mm}}$ = different. arcus $kr = \dfrac{a^3 f\,dx}{\overline{aa-xx}\sqrt{aa\,bb-cc\,xx}}$;
Si jam eg appellatur n, proinde $gl=\sqrt{aa-nn}$ erit

diff. arcus $rl = \dfrac{a\,dn}{\sqrt{aa-nn}}$: Quia itaque $n = \dfrac{abc-afx}{b\sqrt{aa-xx}}$

erit $$dn = \frac{-a^3 f\,dx + abcx\,dx}{\overline{aab-bxx}\sqrt{aa-xx}},$$

et quia $$\sqrt{aa-nn} = \sqrt{\frac{a^4bb-aa\,cc\,xx+2aa\,bc\,fx-aa\,bb\,cc}{aa\,bb-bb\,xx}},$$
erit
$$\frac{a\,dn}{\sqrt{aa-nn}} = \text{different. arcus } lr$$
$$= \frac{-a^3 f\,dx + abc\,x\,dx}{\overline{aa-xx}\sqrt{aa\,bb-cc\,xx+2bcfx-bb\,cc}},$$
ideoque quia different. arcus lr = differ. arcus kr habetur
$$\frac{a^3 f\,dx}{\overline{aa-xx}\sqrt{aa\,bb-cc\,xx}} = \frac{-a^3 f\,dx + abc\,x\,dx}{\overline{aa-xx}\sqrt{aa\,bb-cc\,xx+2bcfx-bb\,cc}}$$
diviso utroque termino per $a\,dx$ et multiplic. per $aa-xx$ erit
$$\frac{aaf}{\sqrt{aa\,bb-ccxx}} = \frac{-aaf+bcx}{\sqrt{aa\,bb-cc\,xx+2bcfx-bb\,cc}},$$
vel multipl. per crucem
$$aaf\sqrt{aa\,bb-cc\,xx+2bcfx-bb\,cc} = \overline{-aaf+bcx}\sqrt{aa\,bb-cc\,xx},$$
sumptisque Quadratis erit $a^6bbff - a^4ffcc\,xx + 2a^4f^3bcx - a^4ffbbcc$
$= a^6bbff - 2a^4b^3fcx + aab^4cc\,xx - a^4ffcc\,xx + 2aafbc^3x^3 - bbc^4x^4$
reducta aequatione ad cyphram et divisa per bc, erit
$$bc^3x^4 - 2aafcc\,x^3 - aab^3cxx + 2a^4bbfx + 2a^4f^3x - a^4ffbc = 0$$
vel quia $ff+bb=cc$ erit divisa aequatione per c,
$$bccx^4 - 2aafcx^3 - aab^3xx + 2a^4fcx - a^4ffb = 0,$$
vel substituto valore ipsius $ff=cc-bb$ erit
$$bccx^4 - 2aafcx^3 - aab^3xx + 2a^4fcx - a^4bcc + a^4b^3 = 0$$
dividatur aequatio per $xx-aa$, et habebitur
$$bcc\,xx - 2aafcx + aabcc - aab^3 = 0,$$
vel (propter $cc-bb=ff$) $bcc\,xx - 2aafcx + aabff = 0$,
quae aequatio, si [28] resolvatur, dabit
$$x = \frac{aaf \pm af\sqrt{aa-bb}}{bc}.$$

Hanc aequationem facillime in simplicissimam proportionem convertere possumus, hoc modo: quia JG est ad JH vel $cadf$, ut sinus totus seu a ad sinum elevationis poli, qui igitur erit $\frac{af}{c}$, proinde $\frac{aaf \pm af\sqrt{aa-bb}}{bc} = \frac{a \pm \sqrt{aa-bb}}{b}$ in sinum elevationis poli; ideoque $b \cdot a \pm \sqrt{aa-bb} :: \text{sin elev. poli} \cdot x$. Vel in terminis Trigonometricis habetur haec proportio: Ut sinus rectus arcus crepuscularis, ad sinum versum ejusdem arcus (ob signum $-$) vel ad sinum versum complementi ad dicos rectos ejusdem arcus (ob signum $+$) ita sinus elevationis poli, ad sinum declinationis solis versus Austrum quaesita. Si declinationis solis innotescit, unica operatione locus ejus in Ecliptica inveniri potest. Sin itaque duo sint crepuscula minima, et quidem utrumque celebratur, cum Sol in Signis Australibus versatur, quorum unum, quod per signum — invenitur, fere in quacunque elevatione poli possibile est; alterum autem cum signo $+$ non nisi in regionibus Aequatori valde vicinis potest contingere; in aliis enim ob majorem elevationem poli evenit, ut declinatio Solis quaesita major evadat, quam ejusdem declinatio maxima, quae est $23\frac{1}{2}$ grad. jmo evenit interdum, ut sinus declinationis quaesitae omnino major evadat, quam Sinus totus, et sic Crepusculum minimum secundarium non solum impossibile, sed plane imaginarium fiat. Utrumque autem crepusculum minimum, cum possibile est, bis in anno contingit, quia Sol bis ad eandem declinationem pervenit. Maxima crepuscula fiunt, cum Sol existit in Tropicis, id quod demonstratu facile est.

[29] NB. Existimo crepusculum brevissimum secundarium per signum $+$ inventum quaesito nequaquam satisfacere, sicuti saepius contingit ut unica tantum aequationis radix Problemati satisfaciat. Nam si duo essent crepuscula minima diversa, necessario inter illa intercederet crepusculum maximum, quod tamen per aequationem nostram inventum non est, utpote quae non nisi ad duas dimensiones pervenit: alias si duo essent crepuscula minima cum intercedente maximo, aequatio ad minimum ad tres dimensiones ascendisset.

Problema XXI.

De inventione puncti flexus Curvarum.

Sunt quaedam Curvae, quae duplicam curvaturam habent, ab initio nempe concavam versus axem, et postmodum convexam ad eundem; vel vice versa convexam ab initio, et sub finem concavam; punctum autem illud, quod duas istas curvaturas separat, vel quod finis prioris et principium posterioris est, vocatur punctum flexis vel recurvationis. Quoties itaque Curva suam curvaturam mutat, tot puncta flexus habebit, quae quo modo in curvis determinanda sint, in praesentiarum unum vel alterum modum trademus.

Primus Modus.

Ex Contemplatione Curvarum patet, quod, quousque Curvae uniformem Curvaturam obtinent, Tangentes crescentibus abscissis continuo a vertice curvae recedant; quam primum autem Curva contrariam curvaturam induit, Tangentes crescentibus abscissis iterum ad verticem accedunt. Hoc inquam cuivis naturam curvaturarum attente consideranti obviam venit; Ex his ergo punctum flexus facillime determinatur. Quia enim Tangens in puncto flexus remotissima est a vertice, erit [30] subtangens minus abscissa, vel abscissa minus Subtangente, omnium possibilium maxima, id est $t-x=m$ vel $x-t=m$, proinde per Methodum de maximis et minimis traditam erit $dt-dx=0$ vel $dx-dt=0$. Ex qua aequatione emergit valor abscissae x, ad quam applicata y determinat punctum flexus quaesitum.

Methodus secundus.

Idem hoc punctum aliter sic inveniri potest: Concipio illud ibi esse, ubi Curva simul est convexa et concava, unum enim aeque est ac alterum, quoniam vero utrumque esse non potest, oportet ut sit recta, id est, neque convexa neque concava (: hoc autem non intelligendum est ac si curvae quaedam portio finita esset recta, sed quod saltem duae particulae infinitae parvae in directum jaceant:). Cum ergo in quavis recta posita dx constante, dy sit quoque constans, et proinde ddy (differentiale differentialis ipsius y) sit $=0$, haberi poterit punctum flexus faciendo $ddy=0$ ex qua aequatione determinabitur abscissa x, simulque punctum quaesitum flexus. Exempla per utrumque modum solvenda rem magis illustrabunt.

Sit igitur ABC (Fig. 23) curva data, cujus natura (posito $AD=x$ $BD=y$) exprimitur per hanc aequationem $axx-yxx-aay=0$ quaeritur punctum flexus B. *Per modum priorem.* Sumantur differentialia aequationis, et habebitur $2ax\,dx-xx\,dy-2xy\,dx-aa\,dy=0$ ideoque $2ax\,dx-2yx\,dx=xx\,dy+aa\,dy$, et proinde $2ax-2yx\cdot xx+aa::dy\cdot dx::y\cdot t$; Reperitur itaque

$$t = \frac{xxy + aay}{2ax - 2yx} = \frac{axx}{2ax - 2yx} = \frac{ax}{2a - 2y} = \left(\text{ob } y = \frac{axx}{aa + xx}\right) \frac{aax + x^3}{2aa}.$$

Ergo
$$x - t = \frac{aax - x^3}{2aa} = \text{Maximo}.$$

Proinde ejus differentiale

$$\frac{aa\,dx - 3xx\,dx}{2aa} = 0.$$

Multipl. per $2aa$ et divis. per dx erit $aa - 3xx = 0$ vel $\frac{1}{3}aa = xx$ et $x = a\sqrt{\frac{1}{3}}$.

[31] *Per modum posteriorem.* Quoniam

$$y = \frac{axx}{aa + xx}$$

sumptis differentialibus erit

$$dy = \frac{2a^3x\,dx}{\square : aa + xx} \quad \text{et} \quad d\,dy = \frac{2a^7dx^2 - 4a^5xx\,dx^2 - 6a^3x^4dx^2}{QQ : aa + xx} = 0$$

vel multiplic. per $QQ : \overline{aa + xx}$ et divid. per $2a^3dx^2$ proveniet $a^4 - 2aa\,xx - 3x^4 = 0$ quae aequatio si dividatur per $aa + xx$, dabit $aa - 3xx = 0$ ut prius.

Hi duo modi in curvis Mechanicis non minus succedunt quam in Geometricis, si modo rite adhibeantur, et ratio inter dy et dx quaeratur.

Sit Ex. gr. ABC (Fig. 24*)) curva ejus naturae, ut diametro AF descripto semicirculo AGF et producta applicata BD ad G, BD sit = arcui AG, quaeritur punctum flexus B.

Per modum 1. Sit $AD = x$ AG vel $BD = y$ $AF = 2a$ erit

$$dy = (\text{ob } be = Gg)\, \frac{a\,dx}{\sqrt{2ax - xx}}.$$

Est autem $dy \cdot dx :: y \cdot t$ ergo

$$t = \frac{y\,dx}{dy} = \frac{y\sqrt{2ax - xx}}{a} \quad \text{et} \quad t - x = \frac{y\sqrt{2ax - xx}}{a} - x = \text{Maximo};$$

Ergo ejus differentiale

$$\frac{2ax\,dy - xx\,dy + ay\,dx - yx\,dx}{a\sqrt{2ax - xx}} - dx = 0$$

vel $2axdy - xx\,dy = yx\,dx - ay\,dx + a\,dx\sqrt{2ax - xx}$ et

$$dy = \frac{yx\,dx - ay\,dx + a\,dx\sqrt{2ax - xx}}{2ax - xx}.$$

Supra autem inventum est

*) In der Zeichnung der Handschrift fehlt der Buchstabe C.

$$dy = \frac{a\,dx}{\sqrt{2ax - xx}} = \frac{yx\,dx - ay\,dx + a\,dx\sqrt{2ax - xx}}{2ax - xx}.$$

Reducta aequatione habetur $yx - ay = 0$ et proinde $x = a$.

Per modum 2. Quia

$$dy = \frac{a\,dx}{\sqrt{2ax - xx}} \text{ erit } ddy = \frac{-aa\,dx^2 + ax\,dx^2}{\overline{2aa - xx}\sqrt{2ax - xx}} = 0.$$

Ideoque $-aa + ax = 0$ et $x = a$.

[32] Sic ad determinandum punctum flexus in Conchoide Nicomedis per methodos quas attulimus, oportet ut Natura Conchoideos habeatur per aequationem, quae relationem explicet inter abscissam et applicatam.

Sit ergo (Fig. 25) $AE = a$ $EF = b$ $AD = x$ $BD = y$ erit $BG = a$ $DE = a - x$. Quia $DE \cdot EF :: BG \cdot GF$ erit

$$GF = \frac{ab}{a - x}. \text{ Ergo } GE = \frac{\sqrt{2abbx - bbxx}}{a - x};$$

Sed $GF \cdot GE :: BF \cdot BD$ id est

$$\frac{ab}{a - x} \cdot \frac{\sqrt{2abbx - bbxx}}{a - x} \text{ vel } a \cdot \sqrt{2ax - xx} :: \frac{aa + ab - ax}{a - x} \cdot y.$$

ideoque

$$y = \frac{b}{a - x}\sqrt{2ax - xx} + \sqrt{2ax - xx}$$

ejusque different.

$$dy = \frac{aab\,dx}{\overline{aa - 2ax + xx}\sqrt{2ax - xx}} + \frac{a\,dx - x\,dx}{\sqrt{2ax - xx}} \cdot dx :: y$$

$$\left(= \frac{\overline{a + b - x}}{a - x}\sqrt{2ax - xx}\right) \cdot t.$$

Erit ergo

$$t = \frac{\overline{ab - bx + aa - 2ax + xx} \text{ in } \overline{2ax - xx}}{\overline{aab + C : a - x}}.$$

Ponatur jam ut Calculus eo facilior evadat $a - x = z$ et erit

$$t = \frac{\overline{bz + zz} \text{ in } \overline{aa - zz}}{aab + z^3} = \frac{aa\,bz + aa\,zz - bz^3 - z^4}{aab + z^3}$$

et $$t - x = t - a + z = \frac{2aa\,bz + aa\,zz - bz^3 - az^3 - a^3b}{aab + z^3} = \text{Maximo}.$$

Ideoque illius differentiale erit

$$= \frac{2a^4bb\,dz + 2a^4bz\,dz - 3aa\,bb\,zz\,dz - 4aa\,bz^3dz - aaz^4dz}{\square : aab + z^3} = 0.$$

Multiplicetur per $\square : aab + z^3$ et dividatur per $aab\,dz + aaz\,dz$, et habebitur $2aab - 3bzz - z^3 = 0$ vel $z^3 + 3bzz - 2aab = 0$. Cujus aequationis Radix z ostendit valorem ipsius $a - x$, vel abscissae ED, ad quam applicata BD transit per punctum flexus B quaesitum. Et sic per Modum priorem.

[33] Per modum posteriorem ita habetur:

Quia $dy = \frac{aab\,dx}{aa - 2ax + xx\sqrt{2ax - xx}} + \frac{a\,dx - x\,dx}{\sqrt{2ax - xx}}$

substituto z loco $a - x$ erit

$$dy = \frac{aab\,dx}{zz\sqrt{aa - zz}} + \frac{z\,dx}{\sqrt{aa - zz}}$$

eorum ergo differentialia

$$\frac{-2a^4bz\,dx + 3aabz^3dz\,dx}{\overline{aaz^4 - z^6}\sqrt{aa - zz}} + \frac{aa\,dz\,dx}{\overline{aa - zz}\sqrt{aa - zz}} = 0.$$

Multiplicata aequatione per $\overline{aaz^4 - z^6\sqrt{aa - zz}}$ et divisa per $aaz\,dz\,dx$, provenit $z^3 + 3bzz - 2aab = 0$ ut ante. Notandum si $a = b$ aequatio resultans $z^3 + 3azz - 2a^3 = 0$ erit plana, dividi enim poterit per $z + a$ et habetur $zz + 2az - 2aa = 0$ proinde $z = -a + \sqrt{3aa} = ED$.

Sit nunc (Fig. 26) ABC alia species Conchoideos, quae talis est, ut Rectangulum inter FG et GB ubique sit aequale Rectangulo inter FE et EA; quaeritur punctum flexus B.

Positis quae superius, quaeratur relatio inter x et y, hoc modo:

$$DF \cdot EF :: BD \cdot GE \quad \text{ergo} \quad GE = \frac{by}{a + b - x},$$

ex quo invenitur

$$GF = \sqrt{\frac{bbyy}{\square : a + b - x} + bb},$$

et quia

$$EF \cdot ED :: GF \cdot GB \quad \text{erit} \quad GB = \frac{a - x}{b}\sqrt{\frac{bbyy}{\square\, a - x + b} + bb}$$

proinde $\square\; FGB = \overline{a - x}$ in $\frac{byy}{\square\, a - b + x} + b = \square\; FEA = ab$.

Ut calculus abbrevietur sit $a - x = z$ et aequatio inventa mutabitur in hanc $zyy + z^3 + 2bzz + bbz = azz + 2abz + abb$; ideoque

$$yy = \frac{azz + 2abz + abb - z^3 - 2bzz - bbz}{z} \quad \text{et} \quad y = \overline{z + b}\sqrt{\frac{a - z}{z}}$$

$$= \sqrt{az - zz} + b\sqrt{\frac{a - z}{z}} \quad \text{ergo} \quad dy = \frac{a\,dz - 2z\,dz}{2\sqrt{az - zz}} - \frac{ab\,dz}{2z\sqrt{az - zz}}$$

$$= (\text{ob } dz = -dx) \frac{-a\,dx + 2z\,dx}{2\sqrt{az - zz}} + \frac{ab\,dx}{2z\sqrt{az - zz}} \cdot dx$$

$$:: y \left(= z + b\sqrt{\frac{a-z}{z}}\right) \cdot t \text{ ergo } t = \frac{-2z^3 + 2azz - 2bzz + 2abz}{-az + 2zz + ab}$$

[34] et

$$t - x = t - a + z = \frac{-azz - 2bzz + aaz + 3abz - aab}{-az + 2zz + ab} = \text{Maximo,}$$

ideoque illius differentiale quod est

$$\frac{\overline{-aa\,zz - 4ab\,zz + 2aa\,bz - 4abbz + 3aa\,bb} \text{ in } + dz}{\square : \overline{-az + 2zz + ab}} = 0.$$

Multipl. per $\square : \overline{az + 2zz + ab}$ et divid. per $az + ab$ in dz, habebitur

$$-az - 4bz + 3ab = 0 \text{ proinde } z = \frac{3ab}{a + 4b}.$$

Per modum posteriorem idem sic reperitur.

$$\text{Quia} \quad dy = \frac{-a\,dx + 2z\,dx}{2\sqrt{az - zz}} + \frac{ab\,dx}{2z\sqrt{az - zz}},$$

eorum ergo differentialia

$$\frac{aa\,dz\,dx}{4az - 4z\sqrt{az - zz}} + \frac{4ab\,zz\,dz\,dx - 3aa\,bz\,dz\,dx}{4az^3 - 4z^4\sqrt{az - zz}} = 0.$$

Multiplicata aequatione per $\overline{4az^3 - z^4}\sqrt{az - zz}$ et divisa per $az\,dz\,dx$ provenit $az + 4bz - 3ab = 0$ et proinde $z = \frac{3ab}{a + 4b} = ED$, ut prius.

Modus tertius inveniendi punctum flexus.

Allata Exempla sufficiunt ad ostendendum, quo pacto methodus inveniendi puncta flexus reduci possit ad methodum de Maximis et Minimis; Interim quoque patet, quod semper necesse sit, ut relatio inter x et y per aequationem habeatur, si punctum flexus quaerendum sit per methodum propositam. Modum nunc ostendemus puncta flexus determinandi ex sola curvarum generatione, absque ut relatio inter x et y quaeratur. In antecessum autem dicam quomodo punctum flexus concipiendum.

[35] Suppono quamlibet curvam compositam ex infinitis lineolis rectis infinite parvis (Fig. 27) *ab*, *bc*, *cd* etc. et tangentem in quocumque puncto *d* esse nihil aliud quam ipsam lineolam *dc* productam ad *m*; Manifestum autem est, quod si curva exterius est convexa, tangens particulae subsequentis *de* exterius cadat, et angulum faciat infinite parvum *ldm*; Si vero curva exterius

est concava, tangens particulae subsequentis interius cadit; punctum itaque flexus ibi erit, ubi Tangens particulae subsequentis neque exterius neque interius cadit et proinde cum Tangente antecedentis particulae coincidit, id est; ubi duae particulae subsequentes ut de, fg in directum jacent.

Hoc intellecto, omnium curvarum, quarum natura non nisi per generationem et per relationem linearum e puncto quodam communi prodeuntium ad alias quascunque innotescit, punctum flexus si quod habent, generali aequatione determinari potest.

Sit enim Curva quaecunque ABC (Fig. 28) punctum flexus habens in B, id quod quaerendum. Ex puncto dato F (ex quo ductae ad curvam lineae ejusdem generationem vel naturam explicant) duci intelligantur lineae FB, Fb, angulum infinite parvum bFB facientes ductisque ad FB, Fb perpendicularibus FD, Fd, agatur ex puncto B tangens BdD, quae (quia B est punctum flexus) erit etiam Tangens in puncto b. Centro F descriptis arculis Be, gd sit FB vel $Fb = z$ FD vel $Fd = t$ $Be = dy$ erit $be = dz$ $gD = dt$; quoniam ang. $BFe = gFd$ erit $FB \cdot Fd :: Be \cdot gd$, ergo $gd = \frac{t\,dy}{z}$ et (ob similitudinem triangulorum beB et gdD) est $be \cdot Be :: gd \cdot gD$, hoc est $dz \cdot dy :: \frac{t\,dy}{z} \cdot dt$ ergo $\frac{t\,dy^2}{z} = dz\,dt$, vel [36] (ob $t \cdot z :: dy \cdot dz$) $\frac{dy^3}{dz} = dz\,dt$ et $dy^3 = dz^2 dt$, ex qua aequatione, quia dy et dt dari possunt in dz, elicitur quid sit z, vel linea FB, proinde cognita FB cognoscitur etiam punctum flexus B.

Sit ex. gr. ABC (Fig. 29) Conchois prima Nicomedis, cujus A vertex F centrum MN Asymptotos; quaeritur punctum flexus B, absque ut relatio inter abscissas et applicatas quaeratur, sed ex sola generatione Conchoideos; Quod nempe ducta utcunque FB intercepta NB semper sit aequalis constanti AM.

Ad hoc faciendum sit $AM = NB = a$, $FM = b$ FB vel $Fb = z$ $be = dz$ $Be = dy$, ducatur NO parallela ipsi Be erit $FN = z - a$ $no = dz$ $NM = \sqrt{zz - 2az + aa - bb}$ ob similit. triangulorum NMF et Non $NM \cdot MF :: no \cdot oN$ ergo $No = \frac{b\,dz}{\sqrt{zz - 2az + aa - bb}}$ et quia $FN \cdot FB :: No \cdot Be$, erit $Be = \frac{bz\,dz}{z - a\sqrt{zz - 2az + aa - bb}} = dy$. Item $be \cdot Be :: bF \cdot t = \frac{bzz}{z - a\sqrt{zz - 2az + aa - bb}}$, et ejus differentiale $-dt$ (NB. sumitur $-dt$, quia crescentibus z ipsae t decrescunt, et proinde different. ipsius t est quantitas negativa)

$$= \frac{-2ab\,z^3dz^3 + 4aab\,zz\,dz^3 - b^3zz\,dz^3 - 2a^3bz\,dz^3 + 2ab^3z\,dz^3}{\overline{zz-2az+aa-bb}\sqrt{zz-2az+aa-bb} \text{ in } \square : \overline{z-a}},$$

Sit itaque aequatio generalis $dy^3 = dz^2dt$ convertitur in hanc

$$\frac{b^3z^3dz^3}{\overline{zz-2az+aa-bb}\sqrt{zz-2az+aa-bb} \text{ in } C : \overline{z-a}}$$

$$= \frac{2abz^3dz - 4aab\,zz\,dz + b^3zz\,dz + 2a^3bz\,dz - 2ab^3z\,dz}{\overline{zz-2az+aa-bb}\sqrt{zz-2az+aa-bb} \text{ in } \square : \overline{z-a}}$$

multiplicata aequatione per $\overline{zz-2az+aa-bb}\sqrt{zz-2az+aa-bb}$ in $C : \overline{z-a}$, et divisa per $bz\,dz^3$, prodibit reducta ad cyphram haec: $2az^3 - 6aa\,zz + 6a^3z - 3a\,bbz - 2a^4 + 2aa\,bb = 0$. Si $a = b$ oritur $2zz - 6az + 3aa = 0$ et $z = \frac{3}{2}a + \sqrt{\frac{3}{4}aa} = FB$.

[37] Esto nunc ABC (Fig. 30) Conchois altera, in qua nempe singula rectangula FNB eidem FMA sunt aequalia, et positis quae prius, sit $FN = x$ erit $no = dx$ $NM = \sqrt{xx-bb}$

$$No = \frac{b\,dx}{\sqrt{xx-bb}}, \quad NB = \frac{ab}{x}, \quad FB \text{ seu } z = \frac{ab+xx}{x}$$

et $$be \text{ seu } dz = \frac{xx\,dx - ab\,dx}{xx},$$

jam quia $FN \cdot FB :: No \cdot Be$ erit Be seu

$$dy = \frac{abb\,dx + bxx\,dx}{xx\sqrt{xx-bb}}$$

et quia $be \cdot eB :: BF \cdot t$ erit

$$t = \frac{aab^3 + 2abb\,xx + bx^4}{\overline{x^3-abx}\sqrt{xx-bb}},$$

sumptisque differentialibus erit

$$-dt = -6abb\,x^6 - b^3x^6 + 5ab^4x^4 - 4aab^3x^4 + 5aab^5xx + 2a^3b^4xx - a^3b^6$$

$$\text{in } \frac{dx}{\overline{xx-bb}\sqrt{xx-bb} \text{ in } \square : \overline{x^3-abx}}.$$

Si itaque in aequatione generali $dy^3 = dz^2dt$ substituantur valores inventi proveniet haec aequatio

$$\frac{dx^3 \text{ in } C : \overline{abb+bxx}}{\overline{x^8-bbx^6}\sqrt{xx-bb}} =$$

$$\frac{6abbx^6+b^3x^6-5ab^4x^4+4aab^3x^4-5aab^5xx-2a^3b^4xx+a^3b^6 \text{ in } dx^3}{\overline{x^8-bbx^6}\sqrt{xx-bb}}.$$

Multiplica aequationem per $\overline{x^8-bbx^6\sqrt{xx-bb}}$, et reductam ad cyphram divide per $\overline{abbx^4+aab^3xx}$, dx^3 et habebitur

$$6xx-8bb-2ab=0 \text{ ideoque } x=\sqrt{\frac{4bb+ab}{3}}=FN.$$

Sit nunc $ABbC$ (Fig. 31) Parabola Spiralis vel Spiralis Parabolica, cujus vertex A, et centrum C, quae talis est naturae, ut centro C per A descripto Circulo, et ducto quocunque radio CN, secante Curvam in B, quadratum BN sit aequale Rectangulo inter arcum AN et constantem aliquam lineam, quae Parameter appellari potest; quaeritur punctum flexus B.

Sit radius CA vel $CN=a$ Parameter $=b$ $CB=z$ ergo $BN=a-z$,

$$\text{arcus } AN=\frac{aa-2az+zz}{b}, \text{ proinde } -Nn=\frac{-2adz+2z\,dz}{b},$$

$Be=dz$, et quia $CN \cdot Ce :: Nn \cdot be$ erit

$$-be=\frac{+2az\,dz-2zz\,dz}{ab}=dy$$

item $Be \cdot be :: BC \cdot t$ erit

$$t=\frac{-2azz+2z^3}{ab} \quad [38] \text{ ergo } -dt=\frac{-4az\,dz+6zz\,dz}{ab},$$

ideoque dy^3 id est

$$\frac{+8a^3z^3dz^3-24aaz^4dz^3+24az^5dz^3-8z^6dz^3}{a^3b^3}=\frac{4az\,dz^3-6zz\,dz^3}{ab}$$

multiplicata aequatione per a^3b^3 et divisa per $2z\,dz^3$, proveniet ad cyphram reducta haec $4z^5-12az^4+12aaz^3-4a^3zz-3aabbz+2a^3bb=0$ cujus aequationis radix ostendit quantitatem CB.

Animadversio.

Caeterum animadvertendum est, quoniam in omnibus curvis punctum flexus eam obtinet proprietatem, ut Tangens in illo puncto Curvam simul secet ita tamen ut angulus sectionis sit dato quovis minor, id est, ut nulla alia recta inter tangentem (vel si mavis secantem) et curvam per punctum flexus duci possit. Quia enim punctum flexus concavae et convexae portioni curvae est commune, et cum Tangens in convexis exterius, in concavis autem interius cadat, manifestum est Tangentem in puncto flexus ab haec parte extra, ab altera vero intra jacere, id est, ipsam

Curvam in ipso puncto secare; quod autem angulus sectionis dato quo vis minor sit patet, quoniam non obstante quod curvam secat, naturam Tangentis ob id non deponit.

[39] De Calculo Integralium.

Vidimus in praecedentibus quomodo quantitatum differentiales inveniendae sunt: nunc vice versa quomodo differentialium Integrales, id est, eae quantitates quarum sunt differentiales, inveniantur, monstrabimus. Et quidem jam ex supra dictis notum est, dx esse differentialem ipsius $x +$ vel $-$ quantitate aliqua constanti; $x\,dx$ differentialem ipsius $\frac{1}{2}\,xx +$ vel $-$ quant. const.; $xx\,dx$ differentialem ipsius $\frac{1}{3}\,x^3 +$ vel $-$ quant. const.; $x^3 dx$ differ. ipsius $\frac{1}{4}\,x^4 +$ vel $-$ quant. const. etiam $a\,dx$ differentialem ipsius ax, etc.

$a\,x\,dx$ . . . . $\frac{1}{2}\,a\,x\,x$, etc.
$a\,x\,x\,dx$. . . . . $\frac{1}{3}\,a\,x^3$, etc.
$a\,x^3 dx$ . . . . $\frac{1}{4}\,a\,x^4$, etc.

Ex quibus Regula formari potest

$a\,x^m\,dx$ differentialis est quantitatis $\frac{a}{m+1}\,x^{m+1}$

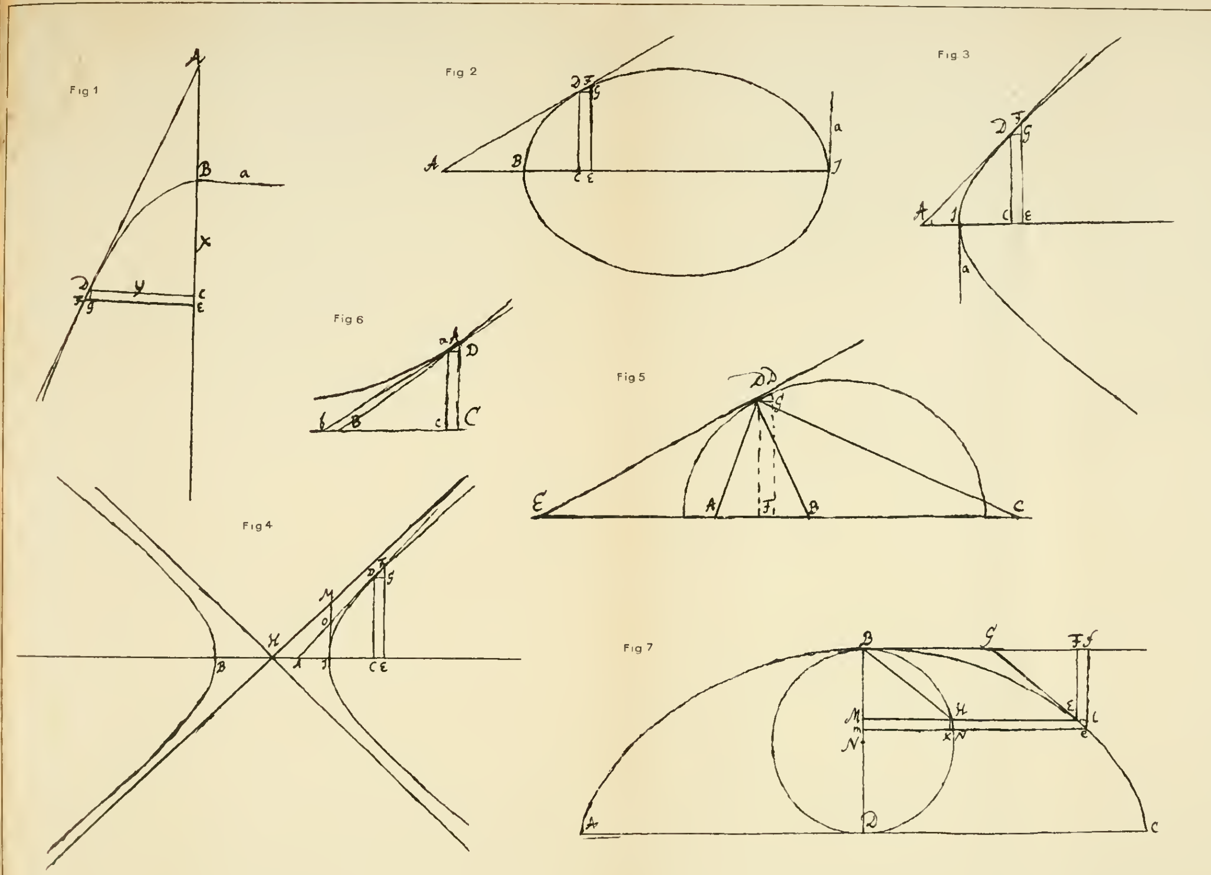
Igitur si alicujus quantitatis differentialis quantitas integralis sumenda sit; ante omnium considerandum est, an quantitas proposita sit productum alicujus differentialis in multiplum suae absolutae ad certam quandam potestatem elevatae: quod signum est ejus Integralem per hanc regulam inveniri posse. Ex. gr. si quantitatis $dy\sqrt{(a+y)}$ integralis invenienda sit, video primo, dy, multiplicatam esse per multiplum suae absolutae $a+y$ ad potestatem $\frac{1}{2}$ elevatae: dein quaero per hanc Regulam ipsius integralem videlicet $\frac{1}{\frac{1}{2}+1}(a+y)^{\frac{1}{2}+1}$, id est, $\frac{2}{3}\,(a+y)\sqrt{(a+y)}$. Sic invenitur integralis ipsius $x\,dx\sqrt[3]{(aa+xx)}$ quae est

$$\frac{\frac{1}{2}}{\frac{1}{3}+1}\,(aa+xx)^{\frac{1}{3}+1} = \frac{3}{8}\,(aa+xx)\sqrt[3]{(aa+xx)};$$

ipsius $dy:\sqrt{(a+y)}$ integralis est $2\sqrt{(a+y)}$; ipsius $dx:x$ integralis est $\frac{1}{0}\,x^0 = \frac{1}{0} =$ Infinito.

Manuskript des Vorworts eingegangen 1. Oktober 1921.

Verhandlungen der Naturf. Gesellschaft in Basel
Bd. XXXIV. Tafel 1.



Graph. Anst. W. Wassermann, Basel

Fig 8

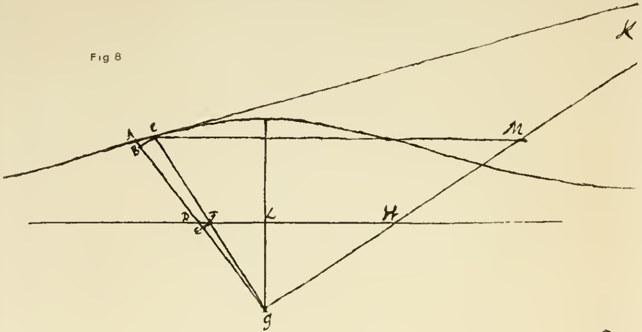


Fig 13

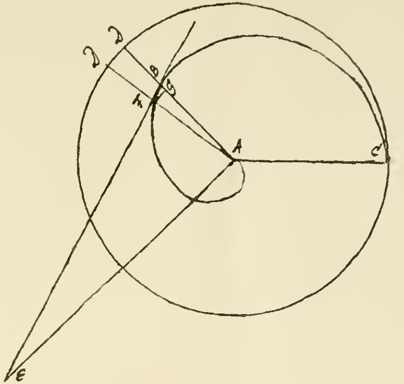


Fig 9

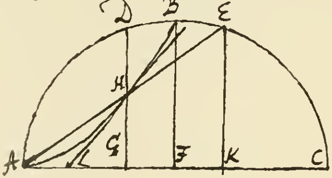


Fig 11

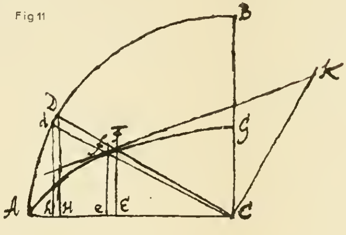


Fig 14

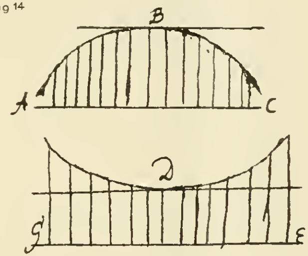


Fig.10

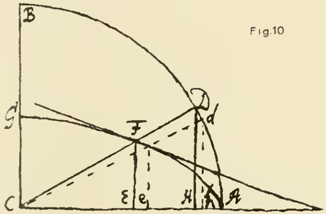


Fig 12

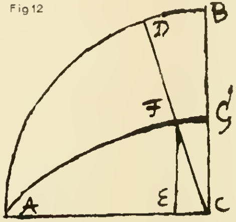


Fig 15

Fig 16

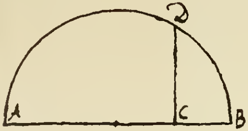


Fig. 17

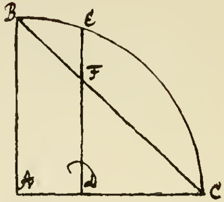


Fig 22a

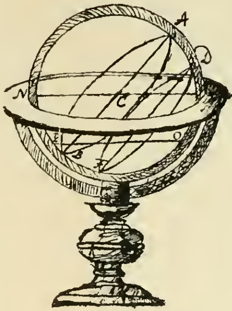


Fig 22b

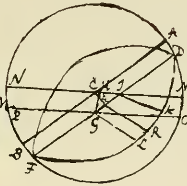


Fig 22c

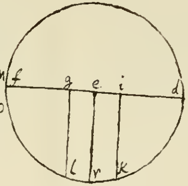


Fig 18

Fig 21

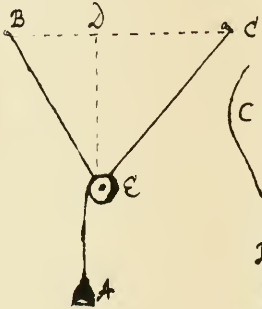


Fig 23

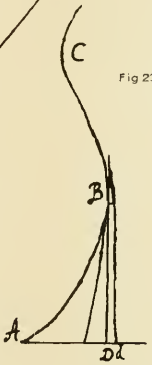


Fig 24

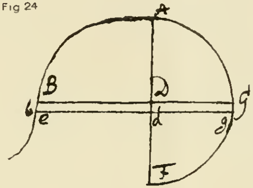


Fig 19

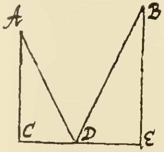


Fig 20

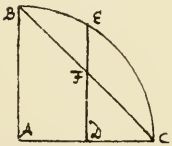
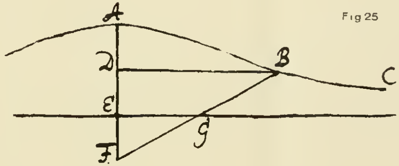


Fig 25



Graph. Anst. W. Wassermann, Basel

Verhandlungen der Naturf. Gesellschaft in Basel
Bd. XXXIV. Tafel IV.

Fig 26

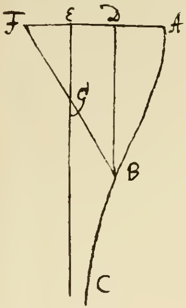


Fig 27

Fig 28

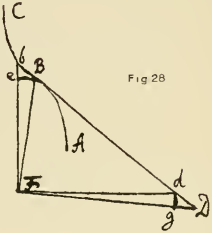


Fig 29

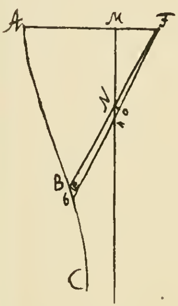


Fig.30

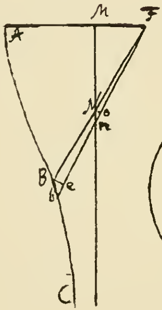
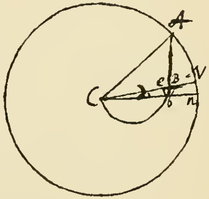


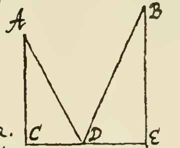
Fig 31



De usu Calculi differentialis in resolvendis Problematibus.

Problema XVII.

In lineâ CE invenire punctum D, à quo si ducantur ad puncta data A et B, lineae DA, DB, ut summa earum sit minima omnium duarum linearum à punctis A et B ad punctum quoddam lineae CE ductarum.



Demittantur perpendiculares $AC=a$ $BE=b$

Sitque $CE=c$ et $CD=x$ Erit $DE=c-x$

$AD=\sqrt{aa+xx}$ et $BD=\sqrt{cc-2cx+xx+bb}$

Et $\sqrt{aa+xx}+\sqrt{cc-2cx+xx+bb}=$ minima.

Ejusque differentialis $\frac{xdx}{\sqrt{aa+xx}}+\frac{xdx-cdx}{\sqrt{cc-2cx+xx+bb}}=0$ Ergo $\frac{x}{\sqrt{aa+xx}}=\frac{c-x}{\sqrt{cc-2cx+xx+bb}}$

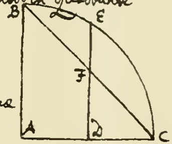
$ccxx-2cx^3+x^4+bbxx=aacc-2aacx+aaxx+ccxx-2cx^3+x^4.$

$bbxx=aacc-2aacx+aaxx$, proinde $bx=ac-ax$ et $bx+ax=ac$

Et denique $x=\frac{ac}{a+b}$.

Problema XVIII.

In Radio AC invenire punctum D, à quo si ducatur perpendicularis DE ad AC, ut abscissa FE inter peripheriam BEC et subtensam BC contenta, sit maxima omnium, quae eodem modo in Quadrante duci possunt.



Sit $AC=a$ $DC=x$ erit et $FD=x$ et $DE=\sqrt{2ax-xx}$, $FE=\sqrt{2ax-xx}-x=$ maxima

Ejusque differentiale $\frac{adx-xdx}{\sqrt{2ax-xx}}-dx=0$

Proinde erit $a-x=\sqrt{2ax-xx}$, $aa-2ax+xx=2ax-xx$.

$xx=\frac{4ax-aa}{2}$ et $x=a\pm\sqrt{\frac{1}{2}aa}$.

Graph. Anst. W. Wassermann, Basel